Mathematics 2433-001H Name (please print) ____________________

Final Examination

December 15, 2005

I. For the quadric surface: $-x^2 + y^2 - z^2 = 4$

(8)

1. In the yz-plane, make a reasonably accurate sketch of the traces with $x = k$, for appropriate ranges of k.

   See graphs download.

2. In the xz-plane, make a reasonably accurate sketch of the traces with $y = k$, for appropriate ranges of k.

   See graphs download.

3. In an xyz-coordinate system, make a reasonably accurate sketch of the quadric surface $-x^2 + y^2 - z^2 = 4$.

   See graphs download.

II. In four xyz-coordinate systems, sketch the following:

(8)

1. The object given in cylindrical coordinates by $r^2 = r$.

   See graphs download.

2. The object given in cylindrical coordinates by $z^2 + r^2 = 1$.

   See graphs download.

3. The object given in spherical coordinates by $3\pi/4 \leq \phi \leq \pi$.

   See graphs download.

4. The object given in spherical coordinates by $\rho = \phi^2$, $0 \leq \phi \leq \pi$.

   See graphs download.

III. In two xy-coordinate systems, sketch the curves given by the following vector equations:

(4) $\vec{r}(t) = (x_0 + at)\vec{\imath} + (y_0 + bt)\vec{\jmath}$ for $-1 \leq t \leq 0$, $\vec{r}(t) = \cosh(t)\vec{\imath} + \sinh(t)\vec{\jmath}$ for all t.

See graphs download.

IV. (12) Regard the helix $x = \cos(2t)$, $y = t$, $z = \sin(2t)$ as a vector-valued function of t.

1. By calculation, verify that the unit tangent vector $\vec{T}(t)$ to the curve is $-2\sin(2t)/\sqrt{5}\,\vec{\imath} + 1/\sqrt{5}\,\vec{\jmath} + 2\cos(2t)/\sqrt{5}\,\vec{k}$.

   Regarded as a vector-valued function of t, the helix is $\vec{r}(t) = \cos(2t)\vec{\imath} + t\vec{\jmath} + \sin(2t)\vec{k}$, and

$$\vec{v}(t) = \vec{r}\,'(t) = -2\sin(2t)\vec{\imath} + \vec{\jmath} + 2\cos(2t)\vec{k}$$
$$\|\, v(t)\,\| = \sqrt{4\sin^2(2t) + 1 + 4\cos^2(2t)} = \sqrt{5}$$
$$\vec{T}(t) = \vec{v}(t)/\|\,\vec{v}(t)\,\| = -2\sin(2t)/\sqrt{5}\,\vec{\imath} + 1/\sqrt{5}\,\vec{\jmath} + 2\cos(2t)/\sqrt{5}\,\vec{k}\ .$$

2. Calculate the unit normal vector $\vec{N}(t)$ to the curve.

$$\vec{n}(t) = \vec{T}\,'(t) = -4\cos(2t)/\sqrt{5}\vec{\imath} - 4\sin(2t)/\sqrt{5}$$
$$\|\,\vec{n}(t)\,\| = \sqrt{16\cos^2(2t)/5 + 16\sin^2(2t)/5} = 4/\sqrt{5}$$
$$\vec{N}(t) = \vec{n}(t)/\|\,\vec{n}(t)\,\| = \cos(2t)\vec{\imath} - \sin(2t)\vec{k}\ .$$

3. Use the formula $\kappa = \dfrac{\|\,\vec{v}(t) \times \vec{a}(t)\,\|}{\|\,\vec{v}(t)\,\|^3}$ to calculate the curvature.

$$\vec{a}(t) = \vec{v}\,'(t) = -4\cos(2t)\vec{\imath} - 4\sin(2t)\vec{k}$$
$$\vec{v}(t) \times \vec{a}(t) = -4\sin(2t)\vec{\imath} - 8\vec{\jmath} + 4\cos(2t)\vec{k}$$
$$\|\,\vec{v}(t) \times \vec{a}(t)\,\| = 4\sqrt{5}$$
$$\|\,\vec{v}(t)\,\| = \sqrt{5}$$
$$\kappa = \frac{4\sqrt{5}}{\sqrt{5}^{\,3}} = \frac{4}{5}$$

4. Use the formula $\kappa = \dfrac{\|\,\vec{T}'(t)\,\|}{\|\,\vec{r}\,'(t)\,\|}$ to calculate the curvature.

$$\kappa = \frac{\|\,\vec{T}'(t)\,\|}{\|\,\vec{r}\,'(t)\,\|} = \frac{\sqrt{16/5}}{\sqrt{5}} = \frac{4}{5}$$

5. Use the formula $\kappa = \left\|\dfrac{d\vec{T}}{ds}\right\|$ to calculate the curvature.

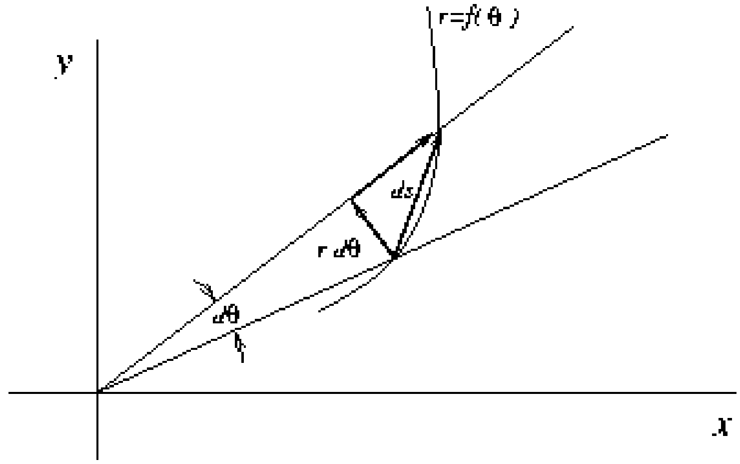
   We have $\dfrac{ds}{dt} = \|\,\vec{v}(t)\,\| = \sqrt{5}$, so

$$\kappa = \left\|\frac{d\vec{T}}{ds}\right\| = \left\|\frac{d\vec{T}}{dt}\Big/\frac{ds}{dt}\right\| = \frac{1}{\sqrt{5}}\left\|\frac{d\vec{T}}{dt}\right\| = \frac{1}{\sqrt{5}}\sqrt{16/5} = \frac{4}{5}$$

V. (3) Bearing in mind that 1 radian is approximately 60 degrees (actually, around 57 degrees), sketch the curve given by these parametric equations in spherical coordinates: $\rho = 1$, $\theta = t/(200\pi)$, $\phi = \pi/2 - \sin(t)$ for $0 \le t \le 200\pi$.

See graphs download.

VI. (4) The figure to the right shows the graph of a polar equation $r = f(\theta)$ in the x-y plane. Use it to determine ds in terms of $d\theta$.



From the Pythagorean Theorem, we have

$$ds^2 = (r\,d\theta)^2 + dr^2 = \left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right) d\theta^{\,2}\ ,$$

so $ds = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}\ d\theta$.

VII. (3) State the Squeeze Principle for limits of sequences.

If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences with $a_n \le b_n \le c_n$ for all (sufficiently large) n, and $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} c_n$ exist and are equal, then $\lim\limits_{n\to\infty} b_n$ exists and equals $\lim\limits_{n\to\infty} a_n$.

VIII. (9) For each of the following series, use standard facts and/or convergence tests to determine whether the series converges or diverges. Give only brief details, but indicate clearly what fact or test you are using, and give at least the key steps in verifying that the test applies.

(i) $\displaystyle\sum \frac{1}{n^2 e^{-n}}$

We have $\displaystyle\lim_{n\to\infty} \frac{1}{n^2e^{-n}} = \lim_{n\to\infty}\frac{e^n}{n^2} = \lim_{x\to\infty}\frac{e^x}{x^2} = \lim_{x\to\infty}\frac{e^x}{2x} = \lim_{x\to\infty}\frac{e^x}{2} = \infty$. Since the terms do not limit to 0, the series diverges.

(ii) $\sum \tan^3(1/\sqrt{n})$

We have

$$\lim \frac{\tan^3(1/\sqrt{n})}{1/n^{3/2}} = \lim\left(\frac{\tan(1/\sqrt{n})}{1/\sqrt{n}}\right)^3 = \lim\left(\frac{1}{\cos(1/\sqrt{n})}\cdot\frac{\sin(1/\sqrt{n})}{1/\sqrt{n}}\right)^3 = \frac{1}{1}\cdot 1 = 1\ .$$

Since $\sum \frac{1}{n^{3/2}}$ converges, the Comparison Test shows that $\sum \tan^3(1/\sqrt{n})$ also converges.

(iii) $\displaystyle\sum \frac{n!}{2\cdot 5\cdot 8\cdots(3n+2)}$

Applying the Ratio Test, we calculate

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\dfrac{(n+1)!}{2\cdot5\cdot8\cdots(3n+2)\cdot(3n+5)}}{\dfrac{n!}{2\cdot5\cdot8\cdots(3n+2)}} = \lim_{n\to\infty}\frac{(n+1)!}{n!}\cdot\frac{1}{3n+5} = \lim_{n\to\infty}\frac{n+1}{3n+5} = \frac{1}{3}\ ,$$

so the Ratio Test shows that this series converges.

IX. (3) A power series of the form $\sum c_n(x+\pi)^n$ converges at $x = 0$ and diverges at $x = \pi$. From this information, what can be determined about its radius of convergence?

The power series is centered at $a = -\pi$, so the radius of convergence is at least $|0-(-\pi)| = \pi$ and no more than $|\pi - (-\pi)| = 2\pi$.

X. (3) Fill in the missing parts of the following argument: Suppose that $\sum |a_n|$ converges. Since $0 \le a_n + |a_n| \le 2|a_n|$, the Comparison Test shows that [fill in]. Since $\sum (a_n + |a_n|)$ and $\sum$ [fill in] converge, it follows that $\sum$ [fill in] converges.

The argument establishes that absolutely convergent series convege. The missing parts are:

1. $\sum a_n + |a_n|$ converges
2. $-|a_n|$ ($|a_n|$ is acceptable)
3. a_n

XI. (12) In the following questions, use the power series $\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

1. For what values of x does the series converge?

Applying the Ratio Test, we calculate

$$\lim \frac{|a_{n+1}|}{|a_n|} = \lim \frac{\dfrac{|x|^{2n+3}}{2n+3}}{\dfrac{|x|^{2n+1}}{2n+1}} = |x|^2 \lim \frac{2n+1}{2n+3} = x^2 \ ,$$

so the Ratio Test shows that this series converges (absolutely) when $x^2 < 1$, that is, for $-1 < x < 1$ and diverges when $x^2 > 1$, that is, when $x < -1$ and $1 < x$. For $x = \pm 1$, the series becomes $\pm \sum \frac{(-1)^n}{2n+1}$, which converges by the Alternating Series Test (but it only converges conditionally, by comparison with the harmonic series). In summary, the series converges for $-1 \le x \le 1$.

2. Calculate $\lim_{x\to 0} \frac{2\tan^{-1}(x/2) - x}{x^3}$.

$$\lim_{x\to 0} \frac{2\tan^{-1}(x/2) - x}{x^3} = \lim_{x\to 0} \frac{2\left(\sum_{n=0}^{\infty} (-1)^n \dfrac{(x/2)^{2n+1}}{2n+1}\right) - x}{x^3} = \lim_{x\to 0} \frac{2\left(\dfrac{x}{2} - \dfrac{x^3}{8\cdot 3} + \dfrac{x^5}{32\cdot 5} - \cdots\right) - x}{x^3}$$

$$= \lim_{x\to 0} \frac{\left(x - \dfrac{x^3}{4\cdot 3} + \dfrac{x^5}{16\cdot 5} - \cdots\right) - x}{x^3} = \lim_{x\to 0} \frac{-\dfrac{x^3}{4\cdot 3} + \dfrac{x^5}{16\cdot 5} - \cdots}{x^3} = \lim_{x\to 0} -\frac{1}{4\cdot 3} + \frac{x^2}{16\cdot 5} - \cdots = -\frac{1}{12} \ .$$

3. Evaluate $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)}$ (Hint: $3^n = \frac{(\sqrt{3})^{2n+1}}{\sqrt{3}}$)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)} = \sqrt{3} \sum_{n=0}^{\infty} (-1)^n \frac{\left(\dfrac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1} = \sqrt{3}\ \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{2\sqrt{3}} \ .$$

4. Find a numerical series whose value is $\int_0^{1/2} \frac{\tan^{-1}(x)}{x}\, dx$ (but do not try to calculate the numerical value of the series).

$$\int_0^{1/2} \frac{\tan^{-1}(x)}{x}\, dx = \int_0^{1/2} \frac{\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}}{x}\, dx = \int_0^{1/2} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}\, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2} \Bigg|_0^{1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)^2}$$

XII. Find an equation for the plane that contains the lines whose vector equations are
(6) $\vec{r}(t) = (11-t)\vec{\imath} + (11+t)\vec{\jmath} + 2t\vec{k}$ and $\vec{r}(t) = (11-2t)\vec{\imath} + (11+t)\vec{\jmath} + t\vec{k}$.

The direction vectors are $\vec{v} = -\vec{\imath} + \vec{\jmath} + 2\vec{k}$ and $\vec{w} = -2\vec{\imath} + \vec{\jmath} + \vec{k}$. A normal vector to the plane is $\vec{v} \times \vec{w} = -\vec{\imath} - 3\vec{\jmath} + \vec{k}$. A point in the plane is when $t = 0$ in either line, that is, $(11, 11, 0)$. An equation for the plane is $(-1)(x-11) + (-3)(y-11) + 1(z-0) = 0$, that is, $x + 3y - z = 44$.

XIII. Tell (without proof or explanation) the convergence behavior of the geometric sequence $\{r^n\}$ for all possible
(5) values of r. At values where it converges, tell the limit. At values where it diverges, tell whether or not it is bounded. At values where it is unbounded, tell whether it diverges to ∞, diverges to $-\infty$, or neither of these.

It diverges when $r < -1$, but neither to ∞ not to $-\infty$.
It diverges but is bounded when $r = -1$ (by alternating between 1 and -1),
It converges to 0 when $-1 < r < 1$.
It converges to 1 when $r = 1$.
It diverges to ∞ when $1 < r$.