Mathematics 2433-001H Name (please print) ______________________

Final Examination

December 15, 2005

I. For the quadric surface: $-x^2 + y^2 - z^2 = 4$
(8)

1. In the yz-plane, make a reasonably accurate sketch of the traces with $x = k$, for appropriate ranges of k.
2. In the xz-plane, make a reasonably accurate sketch of the traces with $y = k$, for appropriate ranges of k.
3. In an xyz-coordinate system, make a reasonably accurate sketch of the quadric surface $-x^2 + y^2 - z^2 = 4$.

II. In four xyz-coordinate systems, sketch the following:
(8)

1. The object given in cylindrical coordinates by $r^2 = r$.
2. The object given in cylindrical coordinates by $z^2 + r^2 = 1$.
3. The object given in spherical coordinates by $3\pi/4 \le \phi \le \pi$.
4. The object given in spherical coordinates by $\rho = \phi^2$, $0 \le \phi \le \pi$.

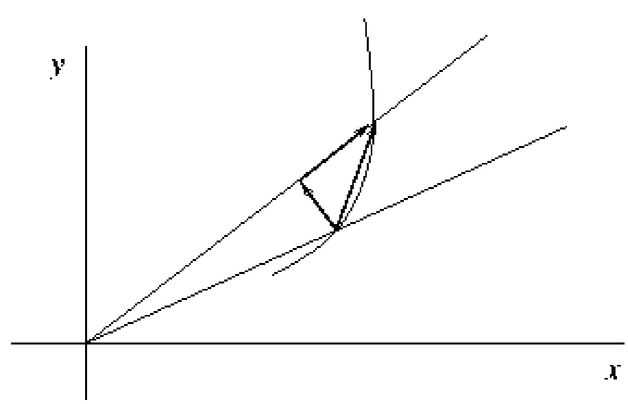
III. In two xy-coordinate systems, sketch the curves given by the following vector equations:
(4) $\vec{r}(t) = (x_0 + at)\vec{\imath} + (y_0 + bt)\vec{\jmath}$ for $-1 \le t \le 0$, $\quad \vec{r}(t) = \cosh(t)\vec{\imath} + \sinh(t)\vec{\jmath}$ for all t.

IV. Regard the helix $x = \cos(2t)$, $y = t$, $z = \sin(2t)$ as a vector-valued function of t.
(12)

1. By calculation, verify that the unit tangent vector $\vec{T}(t)$ to the curve is $-2\sin(2t)/\sqrt{5}\,\vec{\imath} + 1/\sqrt{5}\,\vec{\jmath} + 2\cos(2t)/\sqrt{5}\,\vec{k}$.
2. Calculate the unit normal vector $\vec{N}(t)$ to the curve.
3. Use the formula $\kappa = \dfrac{\|\,\vec{v}(t) \times \vec{a}(t)\,\|}{\|\,\vec{v}(t)\,\|^3}$ to calculate the curvature.
4. Use the formula $\kappa = \dfrac{\|\,\vec{T}'(t)\,\|}{\|\,\vec{r}'(t)\,\|}$ to calculate the curvature.
5. Use the formula $\kappa = \left\|\dfrac{d\vec{T}}{ds}\right\|$ to calculate the curvature.

V. Bearing in mind that 1 radian is approximately 60 degrees (actually, around 57 degrees), sketch the curve
(3) given by these parametric equations in spherical coordinates: $\rho = 1$, $\theta = t/(200\pi)$, $\phi = \pi/2 - \sin(t)$ for $0 \le t \le 200\pi$.

VI. The figure to the right shows the graph of a polar equa-
(4) tion $r = f(\theta)$ in the x-y plane. Use it to determine ds in terms of $d\theta$.



VII. (3) State the Squeeze Principle for limits of sequences.

VIII. (9) For each of the following series, use standard facts and/or convergence tests to determine whether the series converges or diverges. Give only brief details, but indicate clearly what fact or test you are using, and give at least the key steps in verifying that the test applies.

(i) $\sum \dfrac{1}{n^2 e^{-n}}$

(ii) $\sum \tan^3(1/\sqrt{n})$

(iii) $\sum \dfrac{n!}{2 \cdot 5 \cdot 8 \cdots (3n+2)}$

IX. (3) A power series of the form $\sum c_n(x+\pi)^n$ converges at $x = 0$ and diverges at $x = \pi$. From this information, what can be determined about its radius of convergence?

X. (3) Fill in the missing parts of the following argument: Suppose that $\sum |a_n|$ converges. Since $0 \leq a_n + |a_n| \leq 2|a_n|$, the Comparison Test shows that [fill in]. Since $\sum(a_n + |a_n|)$ and $\sum$ [fill in] converge, it follows that $\sum$ [fill in] converges.

XI. (12) In the following questions, use the power series $\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$.

1. For what values of x does the series converge?
2. Calculate $\lim_{x \to 0} \dfrac{2\tan^{-1}(x/2) - x}{x^3}$.
3. Evaluate $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{3^n(2n+1)}$ (Hint: $3^n = \dfrac{(\sqrt{3})^{2n+1}}{\sqrt{3}}$)
4. Find a numerical series whose value is $\int_0^{1/2} \dfrac{\tan^{-1}(x)}{x}\, dx$ (but do not try to calculate the numerical value of the series).

XII. (6) Find an equation for the plane that contains the lines whose vector equations are $\vec{r}(t) = (11 - t)\vec{\imath} + (11 + t)\vec{\jmath} + 2t\vec{k}$ and $\vec{r}(t) = (11 - 2t)\vec{\imath} + (11 + t)\vec{\jmath} + t\vec{k}$.

XIII. (5) Tell (without proof or explanation) the convergence behavior of the geometric sequence $\{r^n\}$ for all possible values of r. At values where it converges, tell the limit. At values where it diverges, tell whether or not it is bounded. At values where it is unbounded, tell whether it diverges to ∞, diverges to $-\infty$, or neither of these.