I. Let $X = \mathbb{R}$ and let $T = \{U \subseteq X \mid \exists M \in \mathbb{R}, (M, \infty) \subseteq U \} \cup \{\emptyset\}$ (where $(M, \infty)$ means $\{r \in \mathbb{R} \mid r > M\}$).

(10) Prove that $T$ is a topology on $X$ (you do not need to worry about special cases involving the empty set).

1. Prove that $T$ is open since $(0, \infty) \subseteq X$. The empty set is open by definition of $T$.

Suppose $\{U_\alpha\}_{\alpha \in A}$ are open sets. Choose a (nonempty) $U_{\alpha_0}$. For some $M$, $(M, \infty) \subseteq U_{\alpha_0}$. Therefore $(M, \infty) \subseteq \bigcup_{\alpha \in A} U_\alpha$, so $\bigcup_{\alpha \in A} U_\alpha$ is also open.

Suppose $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ are open. For each $i$, choose $M_i$ with $(M_i, \infty) \subseteq U_{\alpha_i}$. Put $M$ equal to the maximum of the $M_i$, then $(M, \infty) \subseteq \bigcap_{i=1}^n U_{\alpha_i}$, so $\bigcap_{i=1}^n U_{\alpha_i}$ is open.

2. Prove that with this topology, $X$ is not Hausdorff.

In fact, no two points can have disjoint open neighborhoods. For if $U$ and $V$ were open neighborhoods of two distinct points, then in part 3 of the proof that $T$ is a topology, there exists an interval $(M, \infty) \subseteq U \cap V$, and $U$ and $V$ are not disjoint.

II. Let $S$ be a collection of subsets of a set $X$, such that $X = \bigcup_{S \in S} S$. Define $B = \{S_1 \cap S_2 \cap \cdots \cap S_n \mid S_i \in S\}$, that is, the collection of all subsets of $X$ that are intersections of finitely many elements of $S$. Verify that $B$ is a basis.

By hypothesis, $X = \bigcup_{S \in S} S$. Let $B_1 = S_1 \cap \cdots \cap S_m$ and $B_2 = T_1 \cap \cdots \cap T_n$ be two elements of $B$, and suppose that $x \in B_1 \cap B_2$. Then $x \in S_1 \cap \cdots \cap S_m \cap T_1 \cap \cdots \cap T_n = B_1 \cap B_2$, and $S_1 \cap \cdots \cap S_m \cap T_1 \cap \cdots \cap T_n$ is an element of $B$.

III. Prove that if $B$ is a basis for the topology on a space $X$, and $A \subseteq X$, then $\{B \cap A \mid B \in B\}$ is a basis for the subspace topology on $A$.

It suffices to verify the hypotheses of the Basis Recognition Theorem. Each $B \cap A$ is open in $A$. Suppose that $x \in V$, where $V$ is open in $A$. Since $V$ is open in $A$, there exists an open set $U$ in $X$ with $V = U \cap A$. Since $B$ is a basis for the topology on $X$, there exists $B \in B$ such that $x \in B \subseteq U$. So $x \in B \cap A \subseteq U \cap A = V$.

IV. Prove that there is no countable basis for the lower-limit topology on $\mathbb{R}$.

(10) Given a basis $B$ for the lower-limit topology, we will show that $B$ is uncountable. For each $r \in \mathbb{R}$, choose $B_r \in B$ with $r \in B_r \subseteq [r, r+1)$. If $r \neq s$, say $r < s$, then $r \notin B_s$ since every element of $B_s$ is at least $s$. So all the sets $B_r$ for $r \in R$ are distinct, and $B$ contains uncountably many elements.

V. For each of the following, prove or give a counterexample.

(40) 1. If $f: X \to Y$ is continuous and surjective, and $U$ is an open subset of $X$, then $f(U)$ is an open subset of $Y$.

False. Among many possible examples, take the example $f: [0,1) \to S^1$, where $S^1$ is the unit circle, given by $f(t) = \exp(2\pi it)$. $[0,1/2)$ is open in $[0,1)$ (because it is $[0,1) \cap (-1/2, 1/2)$), but $f([0,1/2))$ is not open in $S^1$, (since $(1,0)$ is a limit point of the complement). Another popular example is the identity function from $[\mathbb{R}, \text{lower limit})$ to $\mathbb{R}$, and $U = [0,1)$.

2. If $f: X \to Y$ is continuous and surjective, and $C$ is a closed subset of $X$, then $f(C)$ is a closed subset of $Y$.
4. If \( X \) is Hausdorff, then each point of \( X \) is a closed subset.

5. Let \( f : X \to Y \) be continuous. If \( x_n \to x \) in \( X \), then \( f(x_n) \to f(x) \) in \( Y \).

6. Let \( f : X \to Y \) be continuous. If \( f(x_n) \to f(x) \) in \( Y \), then \( x_n \to x \) in \( X \).

7. Let \( f : X \to Y \) be continuous and injective. If \( f(x_n) \to f(x) \) in \( Y \), then \( x_n \to x \) in \( X \).

8. If \( T_v \) is a translation of \( \mathbb{R}^2 \) and \( L \) is a linear transformation of \( \mathbb{R}^2 \), then \( L \circ T_v = T_{L(v)} \circ L \).

VI. Let \([0,1]\) be the unit interval in \( \mathbb{R} \). Let \( X \) be a space whose points are closed subsets and having the following property: Given any two disjoint closed subsets \( A \) and \( B \) of \( X \), there exists a continuous function \( f : X \to [0,1] \) such that \( f(A) = \{0\} \) and \( f(B) = \{1\} \). Prove that \( X \) is normal. Hint: \([0,1/4]\) and \((3/4,1]\) are open subsets of \([0,1]\).

The points of \( X \) are closed subsets, by hypothesis. Let \( A \) and \( B \) be disjoint closed subsets of \( X \). By hypothesis, there exists a continuous function \( f : X \to [0,1] \) such that \( f(A) = \{0\} \) and \( f(B) = \{1\} \). Since \([0,1/4]\) and \((3/4,1]\) are open subsets of \([0,1]\) and \( f \) is continuous, \( f^{-1}([0,1/4]) \) and \( f^{-1}((3/4,1]) \) are open in \( X \), and they are disjoint since \([0,1/4]\) and \((3/4,1]\) are. Since \( A \subseteq f^{-1}([0,1/4]) \) and \( B \subseteq f^{-1}((3/4,1]) \), these are disjoint open sets containing \( A \) and \( B \) respectively.

VII. Let \( X \) be the unit circle in the plane, with the usual metric. Prove that every isometry \( J : X \to X \) is surjective.

Suppose that \( J \) is not surjective. Let \( p \) be a point in \( S^1 \) that is not in the image of \( J \), and let \( p' \) be a point that is in the image, say \( p' = J(q') \). If \( d(p,p') = 2 \), then for the unique point \( q \) with \( d(q,q') = 2 \), we must have \( d(J(q), J(q')) = 2 \) and therefore \( J(q) = p \). So we may assume that \( d(p,p') < 2 \). Then, there are two points \( q_1 \) and \( q_2 \) with \( d(q_1,q') = d(q_2,q') = d(p,p') \), and there is one other point \( p_1 \), besides \( p \), with \( d(p_1,p') = d(p,p') \). Since \( p \) is not in the image of \( J \), we can only have \( d(J(q_1), J(q')) = d(J(q_2), J(q')) \) if both \( J(q_1) \) and \( J(q_2) \) equal \( p_1 \), but this would contradict the fact that isometries are injective.