Mathematics 5853 Name (please print) ____________________

Final Examination

December 13, 2004

Instructions: Give brief, clear answers.

I. (10) Let $f\colon X \to Y$ be a function between topological spaces. Prove that f is continuous if and only if for every $x \in X$ and every open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(U) \subseteq V$.

Suppose that f is continuous. Let $x \in X$ and let V be an open neighborhood of $f(x)$. Since f is continuous, $f^{-1}(V)$ is an open neighborhood of x. Taking $U = f^{-1}(V)$, we have $f(U) \subset V$.

Conversely, suppose that for every $x \in X$ and every open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(U) \subseteq V$. Let W be open in Y, and for every $x \in f^{-1}(W)$, choose an open neighborhood U_x with $f(U_x) \subseteq W$. Then $f^{-1}(W) \subseteq \cup_{x\in f^{-1}(W)} U_x \subseteq f^{-1}(W)$, so $f^{-1}(W) = \cup_{x\in f^{-1}(W)} U_x$. Since this is a union of open sets, it is open. Since W was an arbitrary open subset of Y, this proves that f is continuous.

II. (10) Let (X, d_X) and (Y, d_Y) be two metric spaces. A metric $D\colon X \times Y \times X \times Y \to \mathbb{R}$ can be defined by $D((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2)$ (you do not need to verify that D is a metric). Prove that the metric topology for D equals the product topology on $X \times Y$ (Hint: First check that $B_{d_X}(x, \epsilon/2) \times B_{d_Y}(y, \epsilon/2) \subseteq B_D((x, y), \epsilon) \subseteq B_{d_X}(x, \epsilon) \times B_{d_Y}(y, \epsilon)$.)

First we note that if $(x_1, y_1) \in B_{d_X}(x, \epsilon/2) \times B_{d_Y}(y, \epsilon/2)$, then

$$D((x_1, y_1), (x, y)) = d_X(x_1, x) + d_Y(y_1, y) < \epsilon/2 + \epsilon/2 = \epsilon\ ,$$

so $(x_1, y_1) \in B_D((x, y))$. Therefore $B_{d_X}(x, \epsilon/2) \times B_{d_Y}(y, \epsilon/2) \subseteq B_D((x, y), \epsilon)$. Also, if $(x_2, y_2) \in B_D((x, y), \epsilon)$, then

$$d_X(x_2, x) \le d_X(x_2, x) + d_Y(y_2, y) = D((x_2, y_2), (x, y)) < \epsilon\ ,$$

so $x_2 \in B_{d_X}(x, \epsilon)$ and similarly $y_2 \in B_{d_Y}(y, \epsilon)$. This shows that $B_D((x, y), \epsilon) \subseteq B_{d_X}(x, \epsilon) \times B_{d_Y}(y, \epsilon)$.

Now, let $\mathcal{B}$ be the set of ϵ-balls for the D-metric. By definition it generates the D-metric topology. We will verify the conditions of the Basis Recognition Theorem in order to verify that $\mathcal{B}$ it generates the product topology.

To show that the sets $B_D((x, y), \epsilon)$ are open in the product topology, let $(x_1, y_1) \in B_D((x, y), \epsilon)$. Then there exists a δ (in fact, $\delta = \epsilon - D((x_1, y_1), (x, y))$) such that $B_D((x_1, y_1), \delta) \subseteq B_D((x, y), \epsilon)$. So we have $B_{d_X}(x, \delta/2) \times B_{d_Y}(y, \delta/2) \subseteq B_D((x_1, y_1), \delta) \subseteq B_D((x_1, y_1), \epsilon)$. Therefore the sets $B_D((x, y), \epsilon)$ are open in the product topology. For the second condition, let an set U open in the product topology and a point $(x, y) \in U$ be given. There exists a basic open set $B_{d_X}(x, \epsilon_1) \times B_{d_Y}(y, \epsilon_2)$ contained in U. Let ϵ be the minimum of ϵ_1 and ϵ_2. Then we have $(x, y) \in B_D((x, y), \epsilon) \subseteq B_{d_X}(x, \epsilon) \times B_{d_Y}(y, \epsilon) \subseteq U$.

III. (10) Let $X = \prod_{\alpha\in\mathcal{A}} X_\alpha$ be product of nonempty spaces, and suppose that $f\colon Y \to X$ is a function from a space Y into X. Prove that if $\pi_\alpha \circ f$ is continuous for every α, then f is continuous.

Let $\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$ be a basic open set in $\prod_{\alpha\in\mathcal{A}} X_\alpha$. We have

$$\begin{aligned} f^{-1}(\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})) &= f^{-1}(\pi_{\alpha_1}^{-1}(U_{\alpha_1})) \cap \cdots \cap f^{-1}(\pi_{\alpha_n}^{-1}(U_{\alpha_n})) \\ &= (\pi_{\alpha_1} \circ f)^{-1}(U_{\alpha_1}) \cap \cdots \cap (\pi_{\alpha_n} \circ f)^{-1}(U_{\alpha_n})\ . \end{aligned}$$

Since each $\pi_{\alpha_i} \circ f$ is continuous, each $(\pi_{\alpha_i} \circ f)^{-1}(U_{\alpha_i})$ is open, so $(\pi_{\alpha_1} \circ f)^{-1}(U_{\alpha_1})) \cap \cdots \cap (\pi_{\alpha_n} \circ f)^{-1}(U_{\alpha_n})$ is open.

IV. (10) Let X be the real numbers with the cofinite topology.

1. Prove that the integers are a dense subset of X.

   Let U be a nonempty open subset of X. Then $U = \mathbb{R} - \{r_1, \ldots, r_n\}$ for some finite subset $\{r_1, \ldots, r_n\}$ of $\mathbb{R}$. Since this set is finite, it does not contain all of $\mathbb{Z}$, so there exists $n \in \mathbb{Z} \cap U$. Therefore $\mathbb{Z}$ is dense.

2. Prove that X is not second countable.

   Let $\{U_1, U_2, \ldots\}$ be a countable collection of open subsets. We may delete any empty elements to assume that all are nonempty. Each nonempty U_i is of the form $\mathbb{R} - F_i$ for some finite subset. The union $\cup_{i=1}^{\infty} F_i$ is countable, so there exists $r_0 \in \mathbb{R} - \cup_{i=1}^{\infty} F_i$. The set $U = \mathbb{R} - \{r_0\}$ is open, and does not contain any U_i since $r_0 \in U_i$ for every U_i. So U is not a union of any subcollection of $\{U_1, U_2, \ldots\}$, and consequently this collection is not a basis of X.

V. (10) Let $X = \prod_{i=1}^{\infty} \mathbb{R}$ be the product of countably many copies of the real line (where $\mathbb{R}$ has the standard topology and the product has the product topology).

1. State the Tychonoff Theorem.

   A product of compact spaces is compact.

2. Let $A = \{a_n \mid n \in \mathbb{N}\}$ be a set of real numbers, and for each $n \in \mathbb{N}$, let $x_n \in X$ be the point $(a_n, \ldots, a_n, 0, 0, ...)$, where the first n coordinates are a_n and all other coordinates are 0. Suppose that $T\colon X \to \mathbb{R}$ is a continuous function. Prove that if A is a bounded subset of $\mathbb{R}$, then $\{T(x_n) | n \in \mathbb{N}\}$ is a bounded subset of X. Hint: For some M, $A \subset [-M, M]$.

   Since A is a bounded subset of $\mathbb{R}$, there exists a number M so that $A \subset [-M, M]$. Since $[-M, M]$ is compact, the Tychonoff Theorem shows that $\prod_{i=1}^{\infty}[-M, M]$ is a compact subset of X. Therefore $T(\prod_{i=1}^{\infty}[-M, M])$ is a bounded subset of $\mathbb{R}$. Since $T(\prod_{i=1}^{\infty}[-M, M])$ contains every $T(x_n)$, this shows that $\{T(x_n) \mid n \in \mathbb{N}\}$ is a bounded subset of $\mathbb{R}$.

VI. (10) Let (X, d) be a metric space and let A and B be compact subsets of X with $A \cap B = \emptyset$. Prove that there exists $\delta_0 > 0$ such that $d(a, b) \geq \delta_0$ for all $a \in A$ and all $b \in B$ (you may assume that $d\colon X \times X \to \mathbb{R}$ is continuous).

Consider $d|_{A \times B}\colon A \times B \to \mathbb{R}$. It is continuous since it is a restriction of d, and since A and B are compact, $A \times B$ is also compact. Therefore $d|_{A \times B}$ assumes a minimum value, that is, there is a pair $(a_0, b_0) \in A \times B$ with $d(a_0, b_0) \leq d(a, b)$ for all $(a, b) \in A \times B$. Since $A \cap B = \emptyset$, $a_0 \neq b_0$ and therefore $d(a_0, b_0) > 0$. This (or any smaller positive number) is the desired values of δ_0.

VII. Say that a space X is *compactly connected* if for every x and y in X, there exists a compact connected
(20) subset of X that contains x and y. For each of the following statements, prove or give a counterexample.

1. Every compactly connected space is connected.

   Proof: Fix $x_0 \in X$. For each $x \in X$, choose a compact connected set A_x containing x_0 and x. Since all A_x are connected, and thier intersection is nonempty, $X = \cup_{x \in X} A_x$ is connected.

2. Every path-connected space is compactly connected.

   Proof: Given x and y in X, choose a path $\alpha\colon I \to X$ from x to y. Since I is compact and connected, so is the image $\alpha(I)$. Since $\alpha(I)$ contains x and y, this shows that X is compactly connected.

3. The image of a compactly connected space under a continuous map is compactly connected.

   Proof: Let $f\colon X \to Y$ be continuous, with X compactly connected, and choose y_1 and y_2 in $f(X)$. Choose $x_i \in X$ with $f(x_i) = y_i$. Since X is compactly connected, there exists a compact connected set $C \subseteq X$ with $x_1, x_2 \in C$. Then, $f(C)$ is a compact, connected subset of Y containing y_1 and y_2.

4. Every product of compactly connected spaces is compactly connected (you may take as known the fact that an arbitrary product of connected spaces is connected).

   Proof: Let $\prod X_\alpha$ be a product of compactly connected spaces, and let $(x_\alpha), (y_\alpha) \in \prod X_\alpha$. For each α, choose a compact, connected subset $C_\alpha \in X_\alpha$ containing x_α and y_α. Then, $\prod C_\alpha$ is connected, and is compact by the Tychonoff Theorem, so is a compact, connected subset of $\prod X_\alpha$ containing (x_α) and (y_α).

VIII. Recall that a map is called *open* if it takes open sets to open sets.
(10)

1. Prove that a continuous, surjective, open map must be a quotient map.

   Let $f\colon X \to Y$ be continuous, surjective, and open. Let U be a open subset of Y. We must show that U is open in Y if and only if $f^{-1}(U)$ is open in X.
   If U is open in Y, then $f^{-1}(U)$ is open in X since X is continuous. Suppose that $f^{-1}(U)$ is open in X. Since f is an open map, $f(f^{-1}(U))$ is open in Y, and since f is surjective, $f(f^{-1}(U)) = U$.

2. Give an example of a quotient map that is not an open map (you need not verify that the map is a quotient map).

   Define $q\colon [0,1] \to S^1$ by $q(t) = e^{2\pi i t}$. This is a quotient map, but the image of the open set $[0, 1/2)$ is not open in S^1.

IX. State the universal property of quotient maps.
(5)

Let $p\colon X \to Y$ be a quotient map, and let $f\colon Y \to Z$ be a function. The uinversal property says that f is continuous if and only if $f \circ p$ is continuous.

X. (10) Suppose that $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ is an indexed collection of sets, infinitely many of which are noncompact. Prove that $\prod_{\alpha\in\mathcal{A}} X_\alpha$ is not locally compact. (Hint: prove that if a subset C contains a basis element, then there is a continuous surjection from C onto a noncompact factor, and from this, deduce that $\prod_{\alpha\in\mathcal{A}} X_\alpha$ is not locally compact.)

Suppose that $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ is locally compact, and choose a point (x_α) in $\{X_\alpha \mid \alpha \in \mathcal{A}\}$. Then there is a compact subset C containing an open neighborhood $\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$ of (x_α). Since there are inifintely many noncompact factors, we may choose an index β such that X_β is noncompact and β is not equal to one of the α_i. The projection map $\pi_\beta \colon C \to X_\beta$ is surjective, since for any $y \in X_\beta$, the point (y_α) with all $y_\alpha = x_\alpha$ except that $y_\beta = y$ lies in $\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$, and hence in C, and $\pi_\beta((y_\alpha)) = y_\beta = y$. But this is impossible, as C is compact and X_β is noncompact.

XI. (10) Let A be a subset of $\mathbb{R}^5$, which has no limit point in $\mathbb{R}^5$, and let $\sigma\colon A \to \mathbb{R}$ be any function. Prove that there exists a continuous map $f\colon \mathbb{R}^5 \to \mathbb{R}$ with $f|_A = \sigma$.

We know that $\mathbb{R}^5$ normal (since it is metrizable). Now A is closed, since it contains all of its limit points. Also, it has the discrete topology since for every $a \in A$, there exists a neighborhood U of a with $U \cap A = \{a\}$ (otherwise a would be a limit point of A). Since A has the discrete topology, σ is continuous, so the Tietze Extension Theorem implies that there is an extension of σ to a continuous map $f\colon \mathbb{R}^5 \to \mathbb{R}$.

XII. (15) For each of the following, prove or give a counterexample.

1. If $f\colon X \to Y$ is a continuous surjective map between compact Hausdorff spaces, then f is a quotient map.

   Proof: Let C be a subset of Y. If C is closed, then $f^{-1}(C)$ is closed in X since f is continuous. If $f^{-1}(C)$ is closed in X, then it is compact, so $C = f(f^{-1}(C))$ is closed in Y.

2. If a manifold M has nonempty boundary, then M is compact.

   Counterexample: The upper half space $\mathbb{H}$ is a manifold with nonempty boundary, but it is not compact.

3. Let $f\colon X \to Y$ be continuous and injective. If $f(x_n) \to f(x)$ in Y, then $x_n \to x$ in X.

   Counterexample: Let $f\colon [0,1) \to S^1$ be $f(t) = e^{2\pi i t}$. For the sequence $x_n = 1 - \frac{1}{n}$, $f(x_n)$ converges to $(1,0) = f(0)$, but x_n does not converge to 0 in $[0,1)$.