Math 5853 homework solutions

61. Let X be the quotient space obtained from S^1 by identifying all points in the lower half of S^1 to a single point. Prove that X is homeomorphic to S^1. Hint: consider the map $S^1 \to S^1$ that takes $e^{2\pi it}$ to $e^{4\pi it}$ for $0 \leq t \leq 1/2$ and takes $e^{2\pi it}$ to 1 for $1/2 \leq t \leq 1$.

    Define $f\colon S^1 \to S^1$ by sending $e^{2\pi it}$ to $e^{4\pi it}$ for $0 \leq t \leq 1/2$ and $e^{2\pi it}$ to 1 for $1/2 \leq t \leq 1$. To see that f is continuous, let $C_+ = \{(x, y) \in S^1 \mid y \geq 0\}$ and $C_- = \{(x, y) \in S^1 \mid y \leq 0\}$. On C_+, f is the restriction of the complex function $z \mapsto z^2$, so is continuous, and on C_-, f is constant. By gluing on a (locally) finite cover by closed sets, f is continuous. It is surjective, indeed it carries C_+ onto S^1. By inspection, f induces a bijective function $\overline{f}\colon X \to S^1$, and by the universal property of quotient maps, $\overline{f}$ is continuous since f is. Since X is compact (because it is a continuous image of the compact space S^1) and S^1 is Hausdorff, $\overline{f}$ is a homeomorphism.

62. Let X be the quotient space obtained from S^2 by identifying two points whenever they have the same z-coordinate (where as usual, S^2 is regarded as a subset of $\mathbb{R}^3$). Prove that the quotient space is homeomorphic to $[-1, 1]$.

    Define $f\colon S^2 \to [-1, 1]$ by $f(x, y, z) = z$. It is continuous since it is the restriction of a coordinate projection function of $\mathbb{R}^3$. It is surjective since given $z \in [-1, 1]$, $z = f(0, \sqrt{1 - z^2}, z)$. By inspection, f induces a bijective function $\overline{f}\colon X \to [-1, 1]$, and by the universal property of quotient maps, $\overline{f}$ is continuous since f is. Since X is compact (because it is a continuous image of the compact space S^2) and $[-1, 1]$ is Hausdorff, $\overline{f}$ is a homeomorphism.

63. Define the *cone on X*, $C(X)$, to be the quotient space obtained by identifying the subspace $X \times \{1\}$ of $X \times I$ to a point.

    1. The n-ball D^n is defined to be $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 = 1\}$. Prove that $C(S^n)$ is homeomorphic to D^{n+1}. Hint: define $f\colon C(S^n) \to D^{n+1}$ by $f([(x, t)]) = (1 - t)x$.

        Define $g\colon S^n \times I \to D^{n+1}$ by $g(x, t) = (1 - t)x$. It is continuous since it is a composition of projection functions and vector space operations in $\mathbb{R}^{n+1}$. It is surjective since given $v \in D^{n+1}$, either $v = 0$, in which case $v = g(x, 1)$ for any x, or $v \neq 0$, in which case $v = g(\frac{v}{\| v \|}, 1 - \| v \|)$. By inspection, g induces a bijective function $\overline{g}\colon C(S^n) \to D^{n+1}$, and by the universal property of quotient maps, $\overline{g}$ is continuous since g is. Since $C(S^n)$ is compact (because it is a continuous image of the compact space S^n) and D^{n+1} is Hausdorff, $\overline{g}$ is a homeomorphism.

2. Prove that $C(X)$ is path-connected. Deduce that any X is a subspace of a path-connected space.

   Let $([x_0, t_0]) \in C(X)$. Define a path $\alpha\colon I \to X \times I$ by $\alpha(t) = (x_0, t_0 + t(1 - t_0))$. It is continuous since its coordinate functions are continuous. Let $p\colon X \times I \to C(X)$ be the quotient map. Then, $p \circ \alpha$ is a path in $C(X)$ from $[(x_0, t_0)]$ to $([x_0, 1])$. Thus every point in $C(X)$ is in the same path component as the cone point $[(x_0, 1)]$ (for any other $y_0 \in X$, $[(y_0, 1)] = [(x_0, 1)]$), so $C(X)$ is path-connected.

   To deduce that any X imbeds in a path-connected space, it is sufficient to show that X imbeds into $C(X)$. Define $j\colon X \to X \times I$ by $j(x) = (x, 0)$. It is continuous since its coordinate functions are continuous, so $p \circ j\colon X \to C(X)$ is continuous. Also, $p \circ j$ is injective, since $[(x, 0)] = [(y, 0)]$ if and only if $x = y$. To see that $p \circ j$ is an imbedding, it remains to show that it takes open sets in X to open sets in $p \circ j(X)$. Let U be open in X. Now, $U \times [0, 1)$ is open in $X \times I$, and it equals $p^{-1}(p(U \times [0, 1))$, so $p(U \times [0, 1))$ is open in $C(X)$. Since $p \circ j(U) = p(U \times [0, 1)) \cap p \circ j(X)$, $p \circ j(U)$ is open in $p \circ j(X)$.