57. Let $A$ be a closed subset of a normal space $X$. Let $f : A \to \prod_{\alpha \in A} X_{\alpha}$ be continuous, where each $X_{\alpha}$ is homeomorphic either to $\mathbb{R}$ or to a closed interval in $\mathbb{R}$. Prove that $f$ extends to $X$.

For each $\alpha$, the Tietze Extension Theorem gives an extension of $\pi_{\alpha} \circ f : A \to X_{\alpha}$ to $F_{\alpha} : X \to X_{\alpha}$. Define $F : X \to \prod_{\alpha \in A} X_{\alpha}$ by $\pi_{\alpha}(F(a)) = F_{\alpha}(a) = \pi_{\alpha}(f(a))$ for all $\alpha$, so $F(a) = f(a)$.

58. Suppose $X$ is a normal space containing an infinite discrete closed subset $A \subset X$. Prove that there exists a continuous unbounded function from $X$ to $\mathbb{R}$. Deduce that in a compact space, every infinite subset has a limit point in the space. Hint: If $A$ is an infinite subset that has no limit point in $X$, then $A$ contains a countably infinite subset $A_0 = \{a_1, a_2, \ldots\}$ that has no limit point. Such a subset must be a discrete, so $f : A_0 \to \mathbb{R}$ defined by $f(a_n)$ is continuous, and $A_0$ must be closed.

Choose a countable subset $A_0 \subseteq A$, say $A_0 = \{a_1, a_2, \ldots\}$. Since $A$ has the discrete topology, so does $A_0$, so the function $f : A_0 \to \mathbb{R}$ defined by $f(a_n) = n$ is continuous. Also, since $A$ has the discrete topology, $A_0$ is closed in $A$ and therefore closed in $X$. So the Tietze Extension Theorem applies to show that there is an extension $F : X \to \mathbb{R}$ of $f$. Since $f$ is unbounded, so is $F$.

Now, let $X$ be compact and suppose for contradiction that $X$ contains an infinite subset $B$ that has no limit point in $X$. Since $B' = \emptyset$, we have $B = B \cup B' = \overline{B}$, so $B$ is closed in $X$. Moreover, every $b \in B$ has a neighborhood $U$ in $X$ such that $U \cap B = \{b\}$, otherwise $b$ would be a limit point of $B$, so $B$ is a discrete subset of $X$. By the previous argument, this implies that $X$ has an unbounded continuous function, a contradiction to the compactness of $X$.

Jana pointed out that a contradiction can be reached more easily in the second part without depending on the Tietze Extension Theorem: Since $B$ is a closed subset of $X$, it is also compact, but a compact discrete space must be finite.

59. Let $F_n : X \to \mathbb{R}$ be a sequence of functions. Suppose that there are a number $C > 0$ and a number $r \in (0, 1)$ such that $|F_{n+1}(x) - F_n(x)| \leq Cr^n$ for all $x$ in $X$.

1. Tell why $\lim_{n \to \infty} F_n(x)$ exists for each $x \in X$. Hint: observe that the series $\sum_{k=1}^{\infty} F_{k+1}(x) - F_k(x)$ is absolutely convergent.

For each $x$, the series $\sum_{n=1}^{\infty} F_{n+1}(x) - F_n(x)$ converges absolutely by comparison with the geometric series $\sum_{n=1}^{\infty} Cr^n$, so its sequence of partial sums
$s_n = F_{n+1}(x) - F_1(x)$ also converges. But $F_1(x)$ is fixed, so this implies that
the sequence $F_n(x)$ converges.

2. Define $F: X \to \mathbb{R}$ by $F(x) = \lim_{n \to \infty} F_n(x)$. Prove that the sequence $F_n$
converges uniformly to $F$ (that is, for every $\epsilon > 0$ there exists $N$ such that $|F_n(x) - F(x)| < \epsilon$
for all $n \geq N$ and for all $x \in X$).

Given $\epsilon > 0$, choose $N$ so that $\frac{C_{r,N}}{1 - r} < \epsilon$. For each $x$, if $n \geq N$
then
$$|F(x) - F_n(x)| = |\lim_{m \to \infty} F_m(x) - F_n(x)| = \left|\sum_{k=n}^{\infty} F_{k+1}(x) - F_k(x)\right| \leq \left|\sum_{k=n}^{\infty} C_{r,k}\right| =$$
$$\frac{C_{r,n}}{1 - r} < \epsilon,$$
so $F_n$ converges uniformly to $F$.

3. Prove that if $g_n: X \to \mathbb{R}$ is a sequence of continuous functions that converges
uniformly to a function $g: X \to \mathbb{R}$, then $g$ is also continuous.

Given $\epsilon > 0$, choose $N$ so that if $n \geq N$, then $|g(x) - g_n(x)| < \epsilon/3$ for all
$x \in X$. Fix $x_0 \in X$, and choose an open neighborhood $U$ of $x$ so that if $x \in U$
then $|g_N(x) - g_N(x_0)| < \epsilon/3$, which is possible since $g_N$ is continuous (and hence $g_N^{-1}(B(g(x_0), \epsilon/3)$ is open). For any $x \in U$, we have $|g(x) - g(x_0)| \\leq$$|g(x) - g_N(x)| + |g_N(x) - g_N(x_0)| + |g_N(x_0) - g(x_0)| \leq 3(\epsilon/3) = \epsilon,$
establishing the continuity of $g$.

60. Let $A$ be a closed subset of a normal space $X$, and let $f: A \to [a, b]$ be continuous.
Suppose that $f$ extends to a continuous map $G: X \to \mathbb{R}$. Prove that $f$ extends to a
continuous map $F: X \to [a, b]$. Hint: Construct a continuous map $R: \mathbb{R} \to [a, b]$ that
extends the identity on $[a, b]$, and put $F = R \circ G$.

Let $i: [a, b] \to \mathbb{R}$ be the inclusion function. Define $R: \mathbb{R} \to [a, b]$ by $R(x) = b$ if
$x \leq b$, $R(x) = x$ if $a \leq x \leq b$, and $R(x) = b$ if $b \leq x$. This $R$ is continuous (since
its restriction to each of the sets in the finite closed cover $\{(-\infty,], [a, b], [b, \infty)\}$
of $\mathbb{R}$ is continuous), and for $x \in [a, b]$ we have $R \circ i(x) = R(x) = x$, so $R \circ i = id_A$.
For the extension $G: X \to \mathbb{R}$ of $f$, we have on $A$ that $G = i \circ f$, so $R \circ G =
R \circ i \circ f = id_A \circ f = f$, that is, $R \circ G: X \to [a, b]$ is an extension of $f$ as a map
into $[a, b]$.

This last argument is the final part of the proof of the Tietze Extension Theorem; once one
has proven that maps from $A$ to $\mathbb{R}$ extend to $X$, this argument shows that maps from $A$ to
$[a, b]$ extend to $X$. 