Math 5853 homework solutions

57. Let A be a closed subset of a normal space X. Let $f\colon A \to \prod_{\alpha\in\mathcal{A}} X_\alpha$ be continuous, where each X_α is homeomorphic either to $\mathbb{R}$ or to a closed interval in $\mathbb{R}$. Prove that f extends to X.

    For each α, the Tietze Extension Theorem gives an extension of $\pi_\alpha \circ f\colon A \to X_\alpha$ to $F_\alpha\colon X \to X_\alpha$. Define $F\colon X \to \prod_{\alpha\in\mathcal{A}} X_\alpha$ by $\pi_\alpha \circ F = F_\alpha$. By the Fundamental Theorem on Products, F is continuous, and for each $a \in A$, $\pi_\alpha(F(a)) = F_\alpha(a) = \pi_\alpha(f(a))$ for all α, so $F(a) = f(a)$.

58. Suppose X is a normal space containing an infinite discrete closed subset $A \subset X$. Prove that there exists a continuous unbounded function from X to $\mathbb{R}$. Deduce that in a compact space, every infinite subset has a limit point in the space. Hint: If A is an infinite subset that has no limit point in X, then A contains a countably infinite subset $A_0 = \{a_1, a_2, \ldots\}$ that has no limit point. Such a subset must be a discrete, so $f\colon A_0 \to \mathbb{R}$ defined by $f(a_n)$ is continuous, and A_0 must be closed.

    Choose a countable subset $A_0 \subseteq A$, say $A_0 = \{a_1, a_2, \ldots\}$. Since A has the discrete topology, so does A_0, so the function $f\colon A_0 \to \mathbb{R}$ defined by $f(a_n) = n$ is continuous. Also, since A has the discrete topology, A_0 is closed in A and therefore closed in X. So the Tietze Extension Theorem applies to show that there is an extension $F\colon X \to \mathbb{R}$ of f. Since f is unbounded, so is F.

    Now, let X be compact and suppose for contradiction that X contains an infinite subset B that has no limit point in X. Since $B' = \emptyset$, we have $B = B \cup B' = \overline{B}$, so B is closed in X. Moreover, every $b \in B$ has a neighborhood U in X such that $U \cap B = \{b\}$, othersise b would be a limit point of B, so B is a discrete subset of X. By the previous argument, this implies that X has an unbounded continuous function, a contradiction to the compactness of X.

    Jana pointed out that a contradiction can be reached more easily in the second part without depending on the Tietze Extension Theorem: Since B is a closed subset of X, it is also compact, but a compact discrete space must be finite.

59. Let $F_n\colon X \to \mathbb{R}$ be a sequence of functions. Suppose that there are a number $C > 0$ and a number $r \in (0,1)$ such that $|F_{n+1}(x) - F_n(x)| \le Cr^n$ for all x in X.

    1. Tell why $\lim_{n\to\infty} F_n(x)$ exists for each $x \in X$. Hint: observe that the series $\displaystyle\sum_{k=1}^{\infty} F_{k+1}(x) - F_k(x)$ is absolutely convergent.

        For each x, the series $\displaystyle\sum_{n=1}^{\infty} F_{n+1}(x) - F_n(x)$ converges absolutely by comparison with the geometric series $\displaystyle\sum_{n=1}^{\infty} Cr^n$, so its sequence of partial sums

$s_n = F_{n+1}(x) - F_1(x)$ also converges. But $F_1(x)$ is fixed, so this implies that the sequence $F_n(x)$ converges.

2. Define $F\colon X \to \mathbb{R}$ by $F(x) = \lim_{n\to\infty} F_n(x)$. Prove that the sequence F_n converges uniformly to F (that is, for every $\epsilon > 0$ there exists N such that $|F_n(x) - F(x)| < \epsilon$ for all $n \geq N$ and for all $x \in X$).

   Given $\epsilon > 0$, choose N so that $\dfrac{Cr^N}{1-r} < \epsilon$. For each x, if $n \geq N$ then $|F(x) - F_n(x)| = |\lim_{m\to\infty} F_m(x) - F_n(x)| = |\sum_{k=n}^{\infty} F_{k+1}(x) - F_k(x)| \leq |\sum_{k=n}^{\infty} Cr^k| = \dfrac{Cr^n}{1-r} < \epsilon$, so F_n converges uniformly to F.

3. Prove that if $g_n\colon X \to \mathbb{R}$ is a sequence of continuous functions that converges uniformly to a function $g\colon X \to \mathbb{R}$, then g is also continuous.

   Given $\epsilon > 0$, choose N so that if $n \geq N$, then $|g(x) - g_n(x)| < \epsilon/3$ for all $x \in X$. Fix $x_0 \in X$, and choose an open neighborhood U of x so that if $x \in U$, then $|g_N(x) - g_N(x_0)| < \epsilon/3$, which is possible since g_N is continuous (and hence $g_N^{-1}(B(g(x_0), \epsilon/3)$ is open). For any $x \in U$, we have $|g(x) - g(x_0)| \leq |g(x) - g_N(x)| + |g_N(x) - g_N(x_0)| + |g_N(x_0) - g(x_0)| \leq 3(\epsilon/3) = \epsilon$, establishing the continuity of g.

60. Let A be a closed subset of a normal space X, and let $f\colon A \to [a,b]$ be continuous. Suppose that f extends to a continuous map $G\colon X \to \mathbb{R}$. Prove that f extends to a continuous map $F\colon X \to [a,b]$. Hint: Construct a continuous map $R\colon \mathbb{R} \to [a,b]$ that extends the identity on $[a,b]$, and put $F = R \circ G$.

    Let $i\colon [a,b] \to \mathbb{R}$ be the inclusion function. Define $R\colon \mathbb{R} \to [a,b]$ by $R(x) = b$ if $x \leq b$, $R(x) = x$ if $a \leq x \leq b$, and $R(x) = b$ if $b \leq x$. This R is continuous (since its restriction to each of the sets in the finite closed cover $\{(-\infty,], [a,b], [b,\infty)\}$ of $\mathbb{R}$ is continuous), and for $x \in [a,b]$ we have $R \circ i(x) = R(x) = x$, so $R \circ i = id_A$. For the extension $G\colon X \to \mathbb{R}$ of f, we have on A that $G = i \circ f$, so $R \circ G = R \circ i \circ f = id_A \circ f = f$, that is, $R \circ G\colon X \to [a,b]$ is an extension of f as a map into $[a,b]$.

This last argument is the final part of the proof of the Tietze Extension Theorem; once one has proven that maps from A to $\mathbb{R}$ extend to X, this argument shows that maps from A to $[a,b]$ extend to X.