Math 5853 homework solutions

53. Let \((X,d)\) be a metric space. Define \(\overline{d}: X \times X \to \mathbb{R}\) by \(\overline{d}(x,y) = d(x,y)\) when \(d(x,y) \leq 1\) and \(\overline{d}(x,y) = 1\) when \(d(x,y) \geq 1\).

1. Prove that \(\overline{d}\) is a metric on \(X\).

   First, we have \(\overline{d}(x,y) = 0\) if and only if \(d(x,y) = 0\) if and only if \(x = y\).

   For symmetry, if \(d(x,y) < 1\) then \(\overline{d}(x,y) = d(x,y) = d(y,x) = \overline{d}(y,x)\), while

   if \(d(x,y) \geq 1\), then \(d(y,x) \geq 1\) and \(\overline{d}(x,y) = 1 = \overline{d}(y,x)\). For the triangle

   inequality, suppose first that \(d(x,y)\) and \(d(y,z)\) are both less than 1. Then

   \(\overline{d}(x,y) \leq d(x,y) \leq d(x,z) + d(z,y) = \overline{d}(x,z) + \overline{d}(z,y)\). Now, suppose that

   one of \(d(x,y)\) or \(d(y,z)\) is at least 1. Then at least one of \(\overline{d}(x,z)\) or \(\overline{d}(z,y)\)

   equals 1, so \(\overline{d}(x,y) \leq 1 \leq \overline{d}(x,z) + \overline{d}(z,y)\).

2. Observe that \(B_{\overline{d}}(x,\epsilon) = B_d(x,\epsilon)\) when \(\epsilon \leq 1\) and \(B_{\overline{d}}(x,\epsilon) = X\) when \(\epsilon > 1\).

   Suppose first that \(\epsilon \leq 1\). Then \(y \in B_d(x,\epsilon)\) if and only if \(d(x,y) < \epsilon\) if and

   only if \(\overline{d}(x,y) < \epsilon\) if and only if \(y \in B_{\overline{d}}(x,\epsilon)\). Now, suppose that \(\epsilon > 1\). Then

   for all \(y \in X\), \(\overline{d}(x,y) \leq 1 < \epsilon\) so \(y \in B_{\overline{d}}(x,\epsilon)\); that is, \(B_{\overline{d}}(x,\epsilon) = X\).

3. Prove that the metric topology on \(X\) for \(\overline{d}\) equals the metric topology on \(X\) for \(d\). Hint: use the Basis Recognition Theorem to prove that \(\{B_{\overline{d}}(x,\epsilon)\}\) is a basis for the topology on \((X,d)\).

   By part 2, the \(B_{\overline{d}}(x,\epsilon)\) are open sets in the \(d\)-metric topology. Now, suppose

   that \(x \in X\) and that \(U\) is any open neighborhood of \(x\) for the \(d\)-metric topology. Then for some \(\epsilon\), \(B_d(x,\epsilon) \subseteq U\). If \(\epsilon \leq 1\), then \(x \in B_{\overline{d}}(x,\epsilon) = B_d(x,\epsilon) \subseteq U\). If \(\epsilon > 1\), then \(x \in B_{\overline{d}}(x,1/2) = B_d(x,1/2) \subseteq B_d(x,\epsilon) \subseteq U\).

   By the Basis Recognition Theorem, \(\{B_{\overline{d}}(x,\epsilon)\}\) is a basis for the \(d\)-metric topology

   on \(X\). Since by definition it is a basis for the \(\overline{d}\)-metric topology, we conclude

   that the \(\overline{d}\)-metric topology equals the \(d\)-metric topology.

54. Let \(\prod_{\alpha \in A} X_\alpha\) be a product of spaces, and let \(x_n\) be a sequence of points in \(\prod_{\alpha \in A} X_\alpha\). Prove

   that \(x_n\) converges to \(x_0\) if and only if \(\pi_\alpha(x_n)\) converges to \(\pi_\alpha(x_0)\) in \(X_\alpha\) for every \(\alpha\) in \(A\).

   Suppose first that \(x_n \to x_0\) in \(\prod_{\alpha \in A} X_\alpha\). Since each \(\pi_\alpha\) is continuous, and continuous

   functions preserve convergence of sequences, each \(\pi_\alpha(x_n) \to \pi_\alpha(x_0)\). Conversely,

   assume that \(\pi_\alpha(x_n) \to \pi_\alpha(x_0)\) in \(X_\alpha\) for every \(\alpha\) in \(A\). Let \(\cap_{i=1}^k \pi_{\alpha_i}^{-1}(U_{\alpha_i})\) be any

   basic open neighborhood of \(x_0\). For each \(i\) with \(1 \leq i \leq k\), \(\pi_{\alpha_i}(x_n) \to \pi_{\alpha_i}(x_0)\)

   in \(X_{\alpha_i}\), so there exists \(N_i\) such that if \(n > N_i\), then \(\pi_{\alpha_i}(x_n) \in U_{\alpha_i}\). So for

   \(n > \max_{1 \leq i \leq k} \{N_i\}\), \(x_n \in \cap_{i=1}^k \pi_{\alpha_i}^{-1}(U_{\alpha_i})\). Therefore \(x_n \to x_0\).
55. Let \( X = \prod_{\alpha \in A} \mathbb{R} \), where \( A \) is an uncountable set. Let 0 be the point with all coordinates 0, and let \( A = \{(x_{\alpha}) \in \prod_{\alpha \in A} \mathbb{R} | x_{\alpha} \in \{0, 1\} \text{ and } x_{\alpha} = 1 \text{ for all but finitely many } \alpha \} \).

1. Prove that 0 is in \( \overline{A} \).

Let \( \cap_{i=1}^{k} \pi^{-1}_{\alpha_i}(U_{\alpha_i}) \) be any basic open neighborhood of 0. Let \( a \) be the point in \( \prod_{\alpha \in A} \mathbb{R} \) defined by \( \pi_{\alpha_i}(a) = 0 \) for \( 1 \leq i \leq k \), and \( \pi_{\alpha \in A} \alpha_i(a) = 1 \) for \( \alpha \notin \{\alpha_1, \ldots, \alpha_k\} \). Then \( a \in A \), since only finitely many of the \( \pi_{\alpha}(a) \) are 0, so \( a \in A \cap (\cap_{i=1}^{k} \pi^{-1}_{\alpha_i}(U_{\alpha_i})) \). Therefore \( 0 \in \overline{A} \).

2. Prove that there is no sequence of points of \( A \) that converges to 0.

Suppose that \( a_n \) is a sequence of points of \( A \) with \( a_n \to 0 \). For each \( n \), define \( C_n \) to be the set of \( \alpha \) such that \( \pi_{\alpha}(a_n) = 0 \). Each \( C_n \) is finite, so \( \cup_{n=1}^{\infty} C_n \) is countable, so there exists \( \alpha \in A \setminus \cup_{n=1}^{\infty} C_n \). For this \( \alpha \), each \( \pi_{\alpha}(a_n) = 1 \), so \( \pi_{\alpha}(a_n) \to 1 \), contradicting the fact that \( a_n \to 0 \) and therefore \( \pi_{\alpha}(a_n) \to \pi_{\alpha}(0) = 0 \).

3. Deduce that \( X \) is not metrizable.

For a metrizable space, we proved that \( x \in S \) if and only if there exists a sequence of points of \( S \) converging to \( x \). So if \( \prod_{\alpha \in A} \mathbb{R} \) were metrizable, there would have to be a sequence of points of \( A \) converging to 0.

56. Prove that a product of path-connected spaces is path-connected. Hint: Use the Fundamental Theorem for Products.

Let \( X_{\alpha}, \alpha \in A \) be a collection of path-connected spaces, and let \( x, y \in \prod_{\alpha \in A} X_{\alpha} \).

For each \( \alpha \), choose a path \( \gamma_{\alpha} : I \to X_{\alpha} \) with \( \gamma_{\alpha}(0) = \pi_{\alpha}(x) \) and \( \gamma_{\alpha}(1) = \pi_{\alpha}(y) \).

Define \( \gamma : I \to \prod_{\alpha \in A} X_{\alpha} \) by \( \pi_{\alpha} \circ \gamma = \gamma_{\alpha} \) for all \( \alpha \). Since each \( \pi_{\alpha} \circ \gamma \) is continuous, \( \gamma \) is continuous. We have \( \pi_{\alpha} \circ \gamma(0) = \pi_{\alpha}(x) \) for all \( \alpha \), so \( \gamma(0) = x \), and similarly \( \gamma(1) = y \). So we have shown that \( \prod_{\alpha \in A} X_{\alpha} \) is path-connected.