Math 5853 homework

Instructions: All problems should be prepared for presentation at the problem sessions. If a problem has a due date listed, then it should be written up formally and turned in on the due date.

59. (11/30) Let $F_n : X \to \mathbb{R}$ be a sequence of functions. Suppose that there are a number $C > 0$ and a number $r \in (0, 1)$ such that $|F_{n+1}(x) - F_n(x)| \leq Cr^n$ for all $x \in X$.

1. Tell why $\lim_{n \to \infty} F_n(x)$ exists for each $x \in X$. Hint: observe that the series $\sum_{k=1}^{\infty} F_{k+1}(x) - F_k(x)$ is absolutely convergent.

2. Define $F : X \to \mathbb{R}$ by $F(x) = \lim_{n \to \infty} F_n(x)$. Prove that the sequence $F_n$ converges uniformly to $F$ (that is, for every $\epsilon > 0$ there exists $N$ such that $|F_n(x) - F(x)| < \epsilon$ for all $n \geq N$ and for all $x \in X$).

3. Prove that if $g_n : X \to \mathbb{R}$ is a sequence of continuous functions that converges uniformly to a function $g : X \to \mathbb{R}$, then $g$ is also continuous.

60. (11/30) Let $A$ be a closed subset of a normal space $X$, and let $f : A \to [a, b]$ be continuous. Suppose that $f$ extends to a continuous map $G : X \to \mathbb{R}$. Prove that $f$ extends to a continuous map $F : X \to [a, b]$ that extends the identity on $[a, b]$, and put $F = R \circ G$.

61. (12/7) Let $X$ be the quotient space obtained from $S^1$ by identifying all points in the lower half of $S^1$ to a single point. Prove that $X$ is homeomorphic to $S^1$. Hint: consider the map $S^1 \to S^1$ that takes $e^{2\pi it}$ to $e^{4\pi it}$ for $0 \leq t \leq 1/2$ and takes $e^{2\pi it}$ to 1 for $1/2 \leq t \leq 1$.

62. (12/7) Let $X$ be the quotient space obtained from $S^2$ by identifying two points whenever they have the same $z$-coordinate (where as usual, $S^2$ is regarded as a subset of $\mathbb{R}^3$). Prove that the quotient space is homeomorphic to $[-1, 1]$.

63. (12/7) Define the cone on $X$, $C(X)$, to be the quotient space obtained by identifying the subspace $X \times \{1\}$ of $X \times I$ to a point.

1. The $n$-ball $D^n$ is defined to be $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 = 1\}$. Prove that $C(S^n)$ is homeomorphic to $D^{n+1}$. Hint: define $f : C(S^n) \to D^{n+1}$ by $f([x, t]) = (1 - t)x$.

2. Prove that $C(X)$ is path-connected. Deduce that any $X$ is a subspace of a path-connected space.

64. (1/18) The Klein bottle $K$ can be constructed from two annuli $A_1$ and $A_2$ by identifying their boundaries in a certain way. For each of the three descriptions of $K$ discussed in class (two Möbius bands with boundaries identified, the square with certain identifications on its boundary, and $S^1 \times I$ with the two ends identified), make a drawing showing where $A_1$ and $A_2$ appear in $K$. 