Math 5853 homework

Instructions: All problems should be prepared for presentation at the problem sessions. If a problem has a due date listed, then it should be written up formally and turned in on the due date.

53. (11/16) Let (X, d) be a metric space. Define $\overline{d}\colon X \times X \to \mathbb{R}$ by $\overline{d}(x, y) = d(x, y)$ when $d(x, y) \leq 1$ and $\overline{d}(x, y) = 1$ when $d(x, y) \geq 1$.

    1. Prove that $\overline{d}$ is a metric on X.
    2. Observe that $B_{\overline{d}}(x, \epsilon) = B_d(x, \epsilon)$ when $\epsilon \leq 1$ and $B_{\overline{d}}(x, \epsilon) = X$ when $\epsilon > 1$.
    3. Prove that the metric topology on X for $\overline{d}$ equals the metric topology on X for d. Hint: use the Basis Recognition Theorem to prove that $\{B_{\overline{d}}(x, \epsilon)\}$ is a basis for the topology on (X, d).

As a consequence of the previous problem, we can always choose the metric of a metrizable space in such a way that the space has diameter at most 1.

54. (11/16) Let $\prod_{\alpha \in \mathcal{A}} X_\alpha$ be a product of spaces, and let x_n be a sequence of points in $\prod_{\alpha \in \mathcal{A}} X_\alpha$. Prove that x_n converges to x_0 if and only if $\pi_\alpha(x_n)$ converges to $\pi_\alpha(x_0)$ in X_α for every α in $\mathcal{A}$.

55. (11/16) Let $X = \prod_{\alpha \in \mathcal{A}} \mathbb{R}$, where $\mathcal{A}$ is an uncountable set. Let 0 be the point with all coordinates 0, and let $A = \{(x_\alpha) \in \prod_{\alpha \in \mathcal{A}} \mathbb{R} \mid x_\alpha \in \{0, 1\} \text{ and } x_\alpha = 1 \text{ for all but finitely many } \alpha\}$.

    1. Prove that 0 is in $\overline{A}$.
    2. Prove that there is no sequence of points of A that converges to 0.
    3. Deduce that X is not metrizable.

56. (11/16) Prove that a product of path-connected spaces is path-connected. Hint: Use the Fundamental Theorem for Products.

57. (11/30) Let A be a closed subset of a normal space X. Let $f\colon A \to \prod_{\alpha \in \mathcal{A}} X_\alpha$ be continuous, where each X_α is homeomorphic either to $\mathbb{R}$ or to a closed interval in $\mathbb{R}$. Prove that f extends to X.

58. (11/30) Suppose X is a normal space containing an infinite discrete closed subset $A \subset X$. Prove that there exists a continuous unbounded function from X to $\mathbb{R}$. Deduce that in a compact space, every infinite subset has a limit point in the space. Hint: If A is an infinite subset that has no limit point in X, then A contains a countably infinite subset $A_0 = \{a_1, a_2, \ldots\}$ that has no limit point. Such a subset must be a discrete, so $f\colon A_0 \to \mathbb{R}$ defined by $f(a_n)$ is continuous, and A_0 must be closed.