Math 5853 homework

Instructions: All problems should be prepared for presentation at the problem sessions. If a problem has a due date listed, then it should be written up formally and turned in on the due date.

32. (10/5) Let a_0, a_1, a_2 and b_0, b_1, b_2 be two affinely independent sets in $\mathbb{R}^2$. Prove that the formula $M(\sum \lambda_i a_i) = \sum \lambda_i b_i$ in barycentric coordinates defines an affine homeomorphism M, which is the unique affine homeomorphism taking a_i to b_i for $0 \leq i \leq 2$. Hint: This can be done rather easily using facts that we established in class, in particular using our construction of the affine map $T_{a_0} \circ L$ that takes each e_i to a_i, the fact that $(T_{a_0} \circ L)^{-1} = L^{-1} \circ T_{-a_0}$, and the fact that in barycentric coordinates any affine map M takes $\sum \lambda_i p_i$ to $\sum \lambda_i M(p_i)$. For the uniqueness, consider another such affine homeomorphism M' and examine $M^{-1} \circ M'$.

33. (10/5) Let X be the space of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$, with the metric $d(f, g) = \sup_{x\in\mathbb{R}}\{|f(x) - g(x)|\}$.

    1. Verify that this defines a metric on X. (This is easy if one uses the fact that $\sup\{a + b \mid a \in A, b \in B\} \leq \sup A + \sup B$. If you are not familiar with this fact, give a proof of it.)
    2. Prove that (X, d) is not separable. Hint: Let $S = \{f_1, f_2, \ldots\}$ be any countable subset. Construct $f \in X$ so that $|f(n) - f_n(n)| > 2$ for all n, and consider $B(f, 1)$.

34. (10/5) 1.9.26, 1.9.28

35. (10/5) 1.9.55, 1.9.56

36. (10/12) A map $f: X \to Y$ is called an *open map* if it takes open sets to open sets, and is called a *closed map* if it takes closed sets to closed sets. For example, a continuous bijection is a homeomorphism if and only if it is a closed map and an open map.

    1. Give examples of continuous maps from $\mathbb{R}$ to $\mathbb{R}$ that are open but not closed, closed but not open, and neither open nor closed. For the latter two cases, try to give a surjective example.
    2. Prove that a continuous map from a compact space to a Hausdorff space must be closed.
    3. Prove that a projection map from a product $\prod_{i=1}^{n} X_i$ to one of its factors is open, but need not be closed.