Math 5853 homework

Instructions: All problems should be prepared for presentation at the problem sessions. If a problem has a due date listed, then it should be written up formally and turned in on the due date.

29. Verify the following facts about the isometries that generate $\mathrm{Isom}(\mathbb{R}^2)$. The first one is worked here as an example, using the fact that R_θ is linear.

    1. $T_{R_\theta(v)} R_\theta = R_\theta T_v$. Solution: For any $p \in \mathbb{R}^2$, $R_\theta T_v(p) = R_\theta(p+v) = R_\theta(p) + R_\theta(v) = T_{R_\theta(v)} R_\theta(p)$.
    2. $\tau\, T_v = T_{\tau(v)}\, \tau$
    3. $\tau\, R_\theta = R_{-\theta}\, \tau$ (One can regard τ as multiplication by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, R_θ as multiplication by $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ and use matrix multiplication. Or, note that τ is linear and observe geometrically that the two sides have the same geometric effect on the standard basis vectors e_1 and e_2. Or, observe geometrically that both sides of the equation have the same effect on e_1, e_2, and the origin and use the lemma used in class.)

30. (9/28) Adapt the approach we used to analyze the isometries of $\mathbb{R}^2$ to analyze $\mathrm{Isom}(\mathbb{R})$, as follows:

    1. Prove that if an isometry J of $\mathbb{R}$ fixes 0 and 1, then $J = \mathrm{id}$.
    2. Prove that every isometry J of $\mathbb{R}$ can be written uniquely as $T_r\, \tau^\epsilon$, where T_r is translation by r, $\tau(x) = -x$, and ϵ is either 0 or 1.

31. (10/5) Write elements of $\mathbb{R}^2$ as column vectors. Let $a_0 = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$, $a_1 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, and $a_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

    1. Calculate $v_1 = a_1 - a_0$ and $v_2 = a_2 - a_0$ and verify that they are linearly independent by forming the matrix $M = [v_1\ v_2]$ and checking that its determinant is nonzero.
    2. Verify by computation that the affine homeomorphism $T_{a_0} \circ L$, where L is multiplication by M, carries e_i to a_i for $0 \leq i \leq 2$.
    3. Graph the points a_0, a_1, and a_2, and show the following two subsets of $\mathbb{R}^2$: $\lambda_1 = \lambda_2$, $\{\sum \lambda_i a_i \mid 0 \leq \lambda_i \leq 1/2 \text{ for } 0 \leq i \leq 2\}$.