Math 5853 homework

Instructions: All problems should be prepared for presentation at the problem sessions. If a problem has a due date listed, then it should be written up formally and turned in on the due date.

22. (due 9/21) Let $X = (\mathbb{R}, \mathcal{L})$, the reals with the lower-limit topology, and let $Y = \mathbb{R}$, the reals with the standard topology. Prove that a function $f\colon X \to Y$ is continuous if and only if for every $x_0 \in X$, $\lim_{x \to x_0^+} f(x)$ exists and equals $f(x_0)$ (where $\lim_{x \to x_0^+} f(x)$ means the limit as x approaches x_0 from the right).

23. (9/21) Let X and Y be topological spaces with the cofinite topology. State and prove a simple criterion, in terms of the point preimages $f^{-1}(y)$, for a function $f\colon X \to Y$ to be continuous.

24. Prove or give a counterexample: Suppose $X = \cup_{i=1}^{n} S_i$ where each S_i is either an open subset or a closed subset. If $f\colon X \to Y$ is a function whose restriction to each S_i is continuous, then f is continuous.

25. (9/21) Prove that if X is a Hausdorff topological space such that every bijection $f\colon X \to X$ is a homeomorphism, then X has the discrete topology. Hint: Suppose that X has the property, and that some $\{x_0\}$ is not an open set. Choose $y_0 \neq x_0$ and consider the bijection that interchanges x_0 and y_0 and fixes all other points.

26. (9/21) Consider a topological space X, whose points are closed subsets, such that every bijection from X to X is a homeomorphism. Show by example that X need not have the discrete topology.

27. (9/21) Show that if R_θ is not the identity, and v is a vector, then $T_v \circ R_\theta(p) = p$ for some $p \in \mathbb{R}^2$. (This can be proven either algebraically or geometrically, try to find both kinds of proofs.)

28. (9/21) A *dilation* of a metric space (X, d) is a map $f\colon X \to X$ such that for some $k > 0$ and every $x, y \in X$, $d(f(x), f(y)) = k\, d(x, y)$.

    1. Prove that a dilation is continuous and injective.
    2. Prove that a composition of dilations is a dilation.
    3. Prove or give a counterexample: If f_1 and f_2 are dilations with associated constant $k = 2$, and there exists a point $x_0 \in X$ with $f_1(x_0) = f_2(x_0)$, then $f_1 = f_2$.
    4. Let X be the unit circle in $\mathbb{R}^2$, with the standard metric. Prove that every dilation of X is an isometry.