Mathematics 1823-001H Name (please print) ____________________

Examination III

December 2, 2004

Instructions: Give brief, clear answers. It is not expected that you will be able to do all the problems. Just relax and do your best.

I. (5) State the Mean Value Theorem, including its standard hypotheses.

Suppose that $f(x)$ is a function which is continuous for $a \leq x \leq b$, and differentiable for $a < x < b$. Then there exists a number c with $a < c < b$ for which $f(b) - f(a) = f'(c)(b-a)$.

II. (5) For the function $f(x) = \sqrt{x} - x$, find the value of c that satisfies the Mean Value Theorem when $a = 0$ and $b = 3$.

We have $f'(x) = \dfrac{1}{2\sqrt{x}} - 1$, $f(3) = \sqrt{3} - 3$, and $f(0) = 0$, so we seek c with $\left(\dfrac{1}{2\sqrt{c}} - 1\right)(3-0) = \sqrt{3} - 3 - 0$. This gives $\dfrac{1}{2\sqrt{c}} = \dfrac{1}{\sqrt{3}}$, so $\sqrt{c} = \sqrt{3}/2$ and $c = 3/4$.

III. (15) Using the Mean Value Theorem, verify each of the following assertions, assuming that f and g are functions that are differentiable for all x, and that a and b are numbers with $a < b$:

1. If $f'(x) \leq 0$ for all x, then $f(a) \geq f(b)$.

   $f(b) - f(a) = f'(c)(b-a) \leq 0$, since $f'(c) \leq 0$ and $b - a > 0$, so $f(b) \leq f(a)$.

2. If $f'(x) = g'(x)$ for all x, then there is a number C so that $g(x) = f(x) + C$ for all x.

   Taking $a = 0$ (any other fixed number will do) and $b = x$, the Mean Value Theorem applied to the function $g(x) - f(x)$ gives a number c with $(g(x) - f(x)) - (g(0) - f(0)) = (g'(c) - f'(c))(x - 0) = 0$. Letting $C = g(0) - f(0)$, we have $g(x) - f(x) = C$ so $g(x) = f(x) + C$ for all x.

3. If $f''(x) > 0$ for all x, then for $a < x < b$ the graph of $f(x)$ lies above the tangent line to $y = f(x)$ at the point $(a, f(a))$.

   Let $g(x)$ be the difference between $f(x)$ and its tangent line, that is, $g(x) = f(x) - f(a) - f'(a)(x-a)$, so that $g'(x) = f'(x) - f'(a)$. For any x with $a < x < b$, the Mean Value Theorem gives a number c with $a < c < x$ so that $g(x) = g(x) - g(a) = g'(c)(x-a) = (f'(c) - f'(a))(x-a)$, and applying the Mean Value Theorem to f' now gives a number c_1 with $a < c_1 < c < x$ so that $(f'(c) - f'(a))(x-a) = f''(c_1)(c-a)(x-a)$. Since all three of $f''(c_1)$, $c - a$, and $x - a$ are positive, this shows that $g(x) > 0$, that is, $f(x) > f(a) + f'(a)(x-a)$.

IV. (6) State the Extreme Value Theorem. Give an example of a function $f(x)$ defined on $[0,1]$ that has no maximum value.

Let $f(x)$ be a function which is continuous for $a \leq x \leq b$. Then $f(x)$ assume maximum and minimum values on $[a,b]$, that is, there exist numbers c and d in $[a,b]$ so that $f(c) \leq f(x) \leq f(d)$ for all x in $[a,b]$. The conclusion need not hold for a discontinuous function; for example, the range of the function on $[0,1]$ defined by $f(x) = x$ for x in $(0,1)$, $f(0) = f(1) = 1/2$ is the open interval $(0,1)$, so f has neither a maximum nor a minimum value.

V. (6) Define a function $f(x)$ by $f(x) = x^2$ if $x \le 1$, $f(x) = 1$ if $1 \le x \le 2$, and $f(x) = 3 - x$ if $x \ge 2$.

1. Tell all x-values, if any, where f has a local minimum.

   The local minima occur at $x = 0$ and all x with $1 < x < 2$.

2. Tell all x-values, if any, where f has a local maximum.

   The local maxima occur at all x with $1 \le x \le 2$.

VI. (7) Analyze the function $f(x) = \sqrt{\dfrac{x-5}{x}}$, determining its noteworthy features and where they occur, and use this information to sketch the graph of $f(x)$.

The domain is the set of x for which $\dfrac{x-5}{x} \ge 0$, which one determines to be $x \ge 5$ and $x < 0$. We also note that f has a root at $x = 5$, and $\displaystyle\lim_{x\to\infty} \sqrt{\frac{x-5}{x}} = \lim_{x\to\infty} \sqrt{1 - 5/x} = 1$ and similarly $\displaystyle\lim_{x\to-\infty} \sqrt{\frac{x-5}{x}} = 1$, so that $y = 1$ is a horizontal asymptote as $x \to \infty$ and as $x \to -\infty$. Also, $\displaystyle\lim_{x\to 0^-} \sqrt{\frac{x-5}{x}} = \infty$, since near $x = 0$, $\dfrac{x-5}{x}$ is approximately $\dfrac{-5}{x}$. Finally, we calculate that $f'(x) = \dfrac{1}{2}\left(\dfrac{x-5}{x}\right)^{-\frac{1}{2}} \cdot \dfrac{5}{x^2}$, which is positive at all x in the domain, so f is increasing on each interval of its domain. Also, $\displaystyle\lim_{x\to 0^-} f'(x) = \infty$ and $\displaystyle\lim_{x\to 5+} f'(x) = \infty$, so the graph approaches the vertical as $x \to 0^-$ and as $x \to 5^+$. Putting these together gives a graph as shown on the last page below.

VII. (7) Analyze the function $f(x) = \sin(x) - \tan(x)$ for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$, determining its noteworthy features and where they occur, and use this information to sketch the graph of $f(x)$ for these x-values.

An obvious root is $x = 0$, and we also note that $f(x)$ is odd, since both $\sin(x)$ and $\tan(x)$ are odd. We have $f'(x) = \cos(x) - \sec^2(x) = \dfrac{1}{\cos^2(x)}\Big(\cos^3(x) - 1\Big)$, so $f'(x) < 0$ except that $f'(0) = 0$. We also have $\displaystyle\lim_{x\to\pi/2} \sin(x) - \tan(x) = \lim_{x\to\pi/2} \sin(x)\left(1 - \frac{1}{\cos(x)}\right) = -\infty$, since $\displaystyle\lim_{x\to\pi/2} \frac{1}{\cos(x)} = \infty$. Since $f(x)$ is odd, we have $\displaystyle\lim_{x\to-\pi/2} \sin(x) - \tan(x) = \infty$. Putting these together gives a graph as shown on the last page below.

VIII. (8) A box with open top is constructed from a square piece of cardboard, 3 meters on a side, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

Let the length of the portion of the side that is not in the small corner squares be x, so that the length and width of the resulting box are both equal to x. The side of each small square is then $(3-x)/2$, which is the height of the resulting box, so the volume of the box is $V(x) = x^2(3-x)/2 = 3x^2/2 - x^3/2$, where $0 < x < 3$. Since $V'(x) = 3x - 3x^2/2 = 3x(1 - x/2)$, the maximum volume must occur at the unique critical point between 0 and 3, which is $x = 2$. Each of the small squares then has side $1/2$, so the resulting volume is $2^2 \cdot 1/2 = 2$ cubic meters.

IX. (5) A point travels from the origin $(0,0)$ to the point $(1,0)$, by first traveling in a straight line from $(0,0)$ to a point $(x,1)$ at a speed of 3 (units of distance per second), then traveling in a straight line from $(x,1)$ to $(1,0)$ at a speed of 2. Write an expression for the total required time $T(x)$, but do not try to find its minimum value.

The distance from $(0,0)$ to $(x,1)$ is $\sqrt{1+x^2}$, and since distance is speed times time (which is why we measure speed in miles per hour), the time needed for this part of the trip is $\sqrt{1+x^2}/3$. By the distance formula, the distance from $(x,1)$ to $(1,0)$ is $\sqrt{(x-1)^2+1} = \sqrt{x^2-2x+2}$, so the time required for the second part is $\sqrt{x^2-2x+2}/2$, thus $T(x) = \sqrt{1+x^2}/3 + \sqrt{x^2-2x+2}/2$.

X. (5) Suppose that a function $f(x)$ has a horizontal asymptote $y = L$ as $x \to \infty$. Is it necessarily true that $\lim_{x\to\infty} f'(x) = 0$? Either explain why it is true (if you think it is true), or show how it could be false (if you think it is false).

It is false, since although the values of $f(x)$ may be limiting to L, the derivative may be varying wildly if f has small but steep oscillations. A specific example is $f(x) = \sin(x^2)/x$, for which $\lim_{x\to\infty} f(x) = 0$ (by the Squeeze Principle), but $f'(x) = 2\cos(x^2) - \dfrac{\sin(x^2)}{x^2}$; for large x the second term is close to 0, while the first term oscillates between 2 and -2, so $\lim_{x\to\infty} f'(x) = 0$ does not exist.

XI. (6) Recall that $f'(a)$ is the unique number, if such a number exists, for which $f(a+h) = f(a) + \left(f'(a) + \dfrac{\epsilon(h)}{h}\right)h$ with $\lim_{h\to 0} \dfrac{\epsilon(h)}{h} = 0$. Assuming that $f'(a)$ exists and $f'(a) > 0$, verify that there exists $\delta > 0$ so that $f(a) < f(x)$ if $a < x < a + \delta$.

Since $\lim_{h\to 0} \dfrac{\epsilon(h)}{h} = 0$, there exists $\delta > 0$ so that if $0 < |h| < \delta$, then $\left|\dfrac{\epsilon(h)}{h}\right| < f'(a)$. So when $-\delta < h < \delta$, we have $-f'(a) < \dfrac{\epsilon(h)}{h} < f'(a)$, hence $0 < f'(a) + \dfrac{\epsilon(h)}{h}$, and if additionally $0 < h$, then $0 < \left(f'(a) + \dfrac{\epsilon(h)}{h}\right)h$ and $f(a) < f(a) + \left(f'(a) + \dfrac{\epsilon(h)}{h}\right)h = f(a+h)$. Replacing h by $x - a$, we have: if $0 < x - a < \delta$ (that is, $a < x < a + \delta$), then $f(a) < f(x)$.

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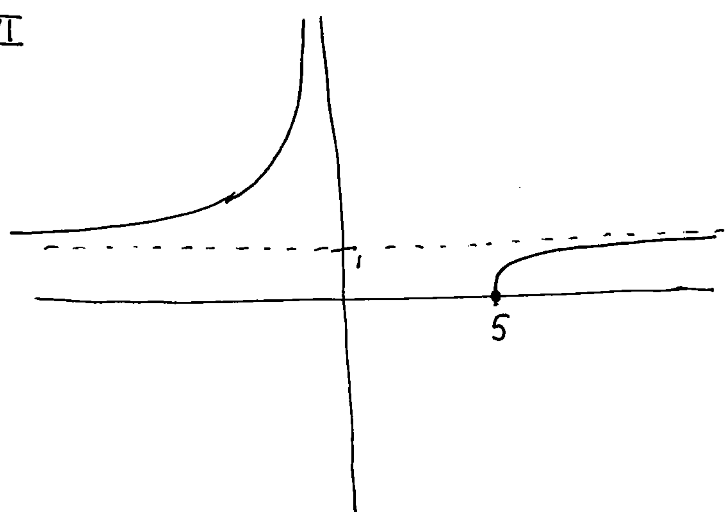

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