Mathematics 2433-003H Name (please print) ______________________

Examination I

September 23, 2004

Instructions: Give brief, clear answers. It is not expected that most people will be able to answer all the questions, just do what you can in 75 minutes.

**I.** (6) Verify that if $f$ is even and $g$ is odd, then $f \circ g$ is even. Verify that if $f$ and $g$ are both odd, then $f \circ g$ is odd.

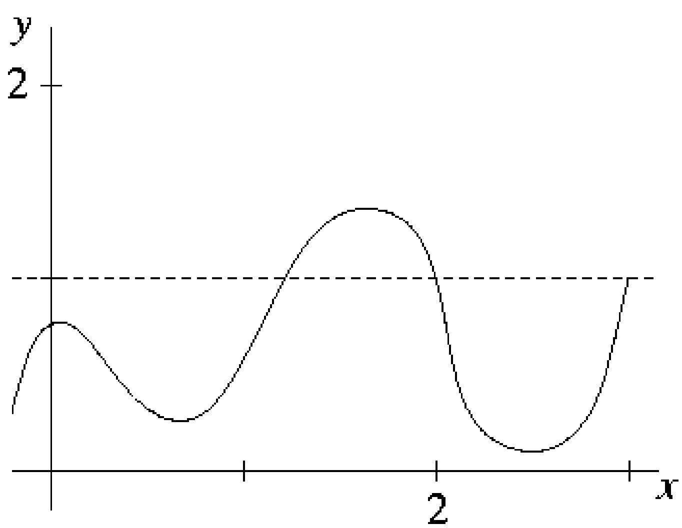
**II.** (6) Write a precise definition of $\lim_{x \to a} f(x) = L$. Write a precise definition of $\lim_{x \to a^-} f(x) = \infty$.

**III.** (8) The graph of a certain function $f(x)$ is the line $y = x + 1$, for $x \le \pi$, and $y = \pi + 1$, for $x \ge \pi$. In separate coordinate systems, sketch the graphs of the following functions: $f(x/2)/2$, $-\frac{1}{2} f(x + \pi)$

**IV.** (8) State the Intermediate Value Theorem. Assuming that the sine function is continuous, use the Intermediate Value Theorem to show that there is a number $x$ between 0 and $\pi/2$ for which $\sin(x) = 1/\sqrt{3}$.

**V.** (4) Give an example of two functions $f(x)$ and $g(x)$ and a point $a$ such that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ do not exist, but $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ does exist.

**VI.** (6) The graph of a certain function $y = f(x)$ is shown at the right, along with the dashed line $y = 1$. The function satisfies $f(2) = 1$. Let $m_{sec}$ be the slope of the secant line from the point $(2, f(2))$ to the point $(2 + h, f(2 + h))$, as a function of $h$. In an $h$-$y$ coordinate system, make a reasonably accurate graph of $y = m_{sec}$ for $-1 \le h \le 1$ (at the very least, correctly indicate where $m_{sec}$ is positive or negative).




**VII.** (8) We established in class that $|\sin(x) - \sin(a)| \le |x - a|$ for any two numbers $x$ and $a$. Use this fact to give an $\epsilon$-$\delta$ proof that $\lim_{x \to a} \sin(x) = \sin(a)$. What property of the sine function does this verify?

**VIII.** (6) Recall that the rate of change of a function $f(x)$ at the $x$-value $a$ is the unique number $m$ for which $f(a+h) = f(a) + mh + \epsilon(h)$ with $\lim_{h \to 0} \dfrac{\epsilon(h)}{h} = 0$ (if such a number $m$ exists). Use this fact together with the calculation $\cos(a+h) = \cos(a)\cos(h) - \sin(a)\sin(h) = \cos(a) - \sin(a)h + \sin(a)(h - \sin(h)) + \cos(a)(\cos(h) - 1)$ and the known limits $\lim_{h \to 0} \dfrac{\sin(h)}{h} = 1$ and $\lim_{h \to 0} \dfrac{\cos(h) - 1}{h} = 0$ to show that the rate of change of $\cos(x)$ at the $x$-value $a$ is $-\sin(a)$.

**IX.** (8) Calculate the limits $\lim_{x\to 7} \dfrac{\sqrt{x+2}-3}{x-7}$ (without using l'Hôpital's rule, of course) and $\lim_{h\to 0} \dfrac{\sin(h)}{2h\cos(h)}$.

**X.** (6) This is a magnified diagram of a small portion of the unit circle, where $a$ and $x$ are two angles that are very close together. Label the angles of the right triangle and the lengths of its two legs in terms of $a$ and $x$. Use these values to obtain an estimate of $\dfrac{\cos(x)-\cos(a)}{x-a}$, and use this estimate to see that the rate of change of the cosine function at the $x$-value $a$ is $-\sin(a)$.