Mathematics 1823-001H Name (please print) ____________________

Final Examination

December 13, 2004

Instructions: Give brief, clear answers. It is not expected that you will be able to do all the problems. Just relax and do your best.

I. (16) Using the Mean Value Theorem, verify each of the following assertions, assuming that f is a function that is differentiable for all x, and that a and b are numbers with $a < b$:

1. If $f'(x) \geq 0$ for all x, then $f(a) \leq f(b)$.

   $f(b) - f(a) = f'(c)(b-a) \geq 0$, since $f'(c) \geq 0$ and $b - a > 0$, so $f(b) \geq f(a)$.

2. If $f'(x) \geq 1/2$ for $a \leq x \leq b$, and $f(3) = 7$, then $f(7) \geq 9$.

   $f(7) - 7 = f(7) - f(3) = f'(c)(7-3) \geq 1/2 \cdot 4 = 2$, so $f(7) \geq 9$.

3. $a - \cos(a) \leq b - \cos(b)$.

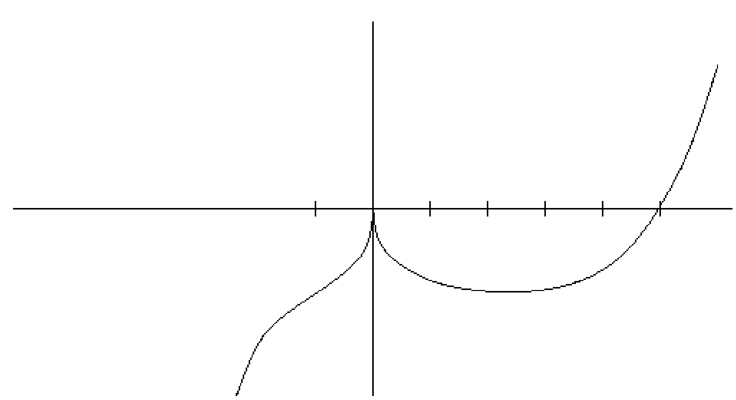
   $\cos(b) - \cos(a) = \cos'(c)(b-a) \leq b - a$, so $a - \cos(a) \leq b - \cos(b)$.
   Or, one can put $f(x) = x - \cos(x)$, note that $f'(x) = 1 + \sin(x) \geq 0$, and quote part 1.

4. If $f''(x) < 0$ for all x, then for $a < x < b$ the graph of $f(x)$ lies below the tangent line to $y = f(x)$ at the point $(a, f(a))$.

   Let $g(x)$ be the difference between $f(x)$ and its tangent line, that is, $g(x) = f(x) - f(a) - f'(a)(x-a)$, so that $g(a) = 0$ and $g'(x) = f'(x) - f'(a)$. For any x with $a < x < b$, the Mean Value Theorem gives a number c with $a < c < x$ so that $g(x) = g(x) - g(a) = g'(c)(x-a) = (f'(c) - f'(a))(x-a)$, and applying the Mean Value Theorem to f' now gives a number c_1 with $a < c_1 < c < x$ so that $(f'(c) - f'(a))(x-a) = f''(c_1)(c-a)(x-a)$. Since $f''(c_1) < 0$ and $c - a$ and $x - a$ are positive, this shows that $g(x) < 0$, that is, $f(x) < f(a) + f'(a)(x-a)$.

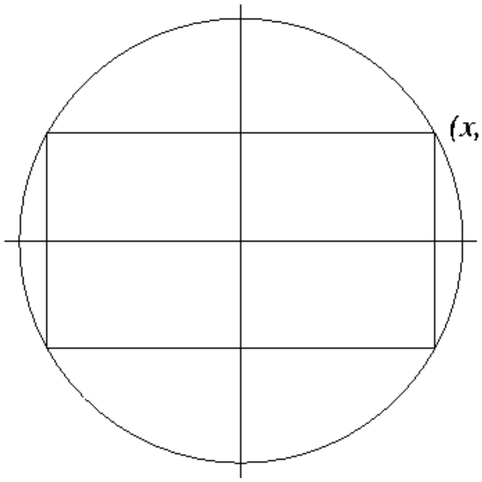
II. (6) Analyze the function $f(x) = x^{5/3} - 5x^{2/3}$, determining its noteworthy features and where they occur, and use this information to sketch the graph of $f(x)$.

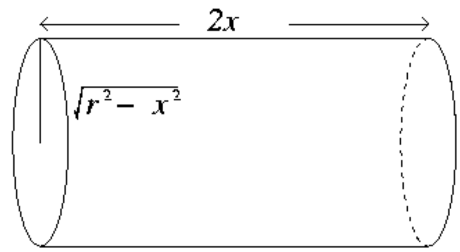
$f(x)$ is defined for all x, and writing $f(x) = x^{2/3}(x-5)$ shows that $f(x)$ has roots at 0 and 5, and that $f(x)$ is negative for $x < 5$ (and $x \neq 0$) and positive for $x > 5$. We calculate $f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3}$, which is defined for all x except $x = 0$. Writing $f'(x) = \frac{5}{3}x^{2/3}(1 - 2/x)$ shows that the only critical point other than $x = 0$ is $x = 2$, and moreover that $f'(x)$ is positive when $x < 0$, negative for $0 < x < 2$, and positive when $x > 2$. The value at the critical point is $f(2) = -3(4)^{1/3}$. For the second derivative, we have $f''(x) = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}(x+1)$, which changes sign at $x = -1$, is negative for $x < -1$, and positive for $x > -1$ (and $x \neq 0$). In particular, $f(x)$ has an inflection point at $x = -1$, where $f(-1) = -6$ and $f'(-1) = 5$. Near $x = 0$, $f'(x) \approx \frac{5}{3}x^{-2/3}$, so $\lim_{x \to 0^-} f'(x) = \infty$ and $\lim_{x \to 0^+} f'(x) = \infty$. Putting these together produces a graph of $f(x)$ like the one shown here:



III. (8) A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

A cross-section of the cylinder in the sphere, and the cylinder determined by a given x-value, look like this:





Using $V = \pi R^2 H$ gives $V(x) = 2\pi(r^2 - x^2)x = 2\pi r^2 x - 2\pi x^3$, $0 \le x \le r$. Since $V'(x) = 2\pi r^2 - 6\pi x^2 = 2\pi(r^2 - 3x^2) = 0$ only when $x = r/\sqrt{3}$, the largest possible volume is $V(r/\sqrt{3}) = \frac{4\pi r^3}{3\sqrt{3}}$.

IV. (5) A function $f(x)$ is differentiable for $x > 0$, and $\lim_{x \to 0^+} f(x) = \infty$. Is it necessarily true that $\lim_{x \to 0^+} f'(x) = -\infty$? Either explain why it is true (if you think it is true), or show how it could be false (if you think it is false).

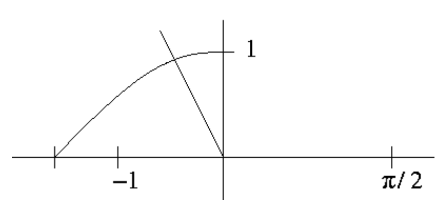
It is false, because $f(x)$ could be oscillating, and thereby producing large variation in $f'(x)$, even though its values are limiting to ∞. An explicit example would be $f(x) = \dfrac{1}{x} + \sin\left(\dfrac{1}{x}\right)$. Since $f(x) \ge \dfrac{1}{x} - 1$ for all $x > 0$, we have $\lim_{x \to 0^+} f(x) = \infty$. But $f'(x) = -\dfrac{1}{x^2} - \cos\left(\dfrac{1}{x}\right)\dfrac{1}{x^2} = -\dfrac{1}{x^2}(\cos\left(\dfrac{1}{x}\right) + 1)$, whose values oscillate between 0 and $-\dfrac{2}{x^2}$ as $x \to 0^+$, so $\lim_{x \to 0^+} f'(x)$ does not exist.

V. (6) The function $x^2 + \sin(x)$ has a unique absolute minimum value at the point where its derivative equals 0 (since its second derivative $2 - \sin(x)$ is always positive). Using Newton's method, set up an explicit iteration that one would use to calculate the location of this minimum value. Graphically estimate a reasonable starting value x_1 for the iteration, but do not try to carry out the iteration computationally.

The derivative is $2x + \cos(x)$, so the iteration is

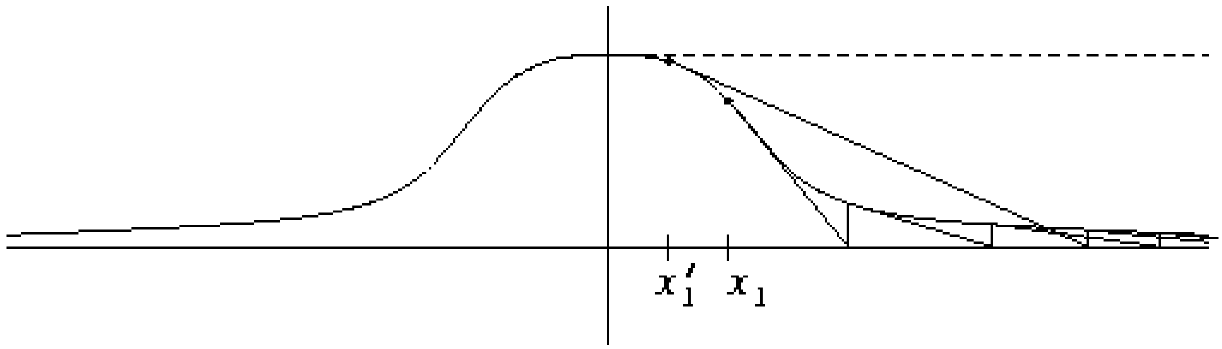
$$x_{n+1} = x_n - \frac{2x_n + \cos(x_n)}{2 - \sin(x_n)} = \frac{\cos(x_n) + x_n \sin(x_n)}{\sin(x_n) - 2}\ .$$

The root of the derivative is where $\cos(x) = -2x$, that is, at the point where the graphs of $\cos(x)$ and $-2x$ cross. The crude graph at the right suggests a starting value of $x_1 = -1/2$, or perhaps $x_1 = -0.4$.



VI. (6) Make a quick sketch of the function $f(x) = \dfrac{1}{x^2+1}$. Using the graph and the geometric interpretation of Newton's method (i. e. not by numerical computation), explain what would happen to the values x_n if you started the iteration of Newton's method at a number $x_1 > 0$. Similarly, use it to explain what would happen to the values x_n if you started the iteration of Newton's method at a number $x_1 < 0$, and what would happen if you started at $x_1 = 0$.

As seen in the graph below, when we start at a positive number x_1, the interates move to the right (in fact, they limit to ∞). If we start very close to 0, as at the number x_1', x_2 will be a larger positive number, and as for x_1, the interates continue moving to the right. When $x_1 < 0$, the iterates will move to $-\infty$, as this is just the mirror image of the case $x_1 > 0$. For $x_1 = 0$, the tangent line is the horizontal dotted line, so x_2 will be undefined.



VII. (8) Use antiderivatives to find all functions $f(x)$ satisfying each of the following:

1. $f''(x) = \sin(x)$.

   $f'(x) = -\cos(x) + C$, so $f(x) = -\sin(x) + Cx + C_1$, where C and C_1 are arbitrary constants.

2. $f''(x) = \sin(x)$ and $f'(\pi/2) = 2$.

   $f'(x) = -\cos(x) + C$, giving $2 = f'(\pi/2) = -0 + C$, so $C = 2$ and $f'(x) = -\cos(x) + 2$. We then find that $f(x) = -\sin(x) + 2x + C_1$, where C_1 is an arbitrary constant.

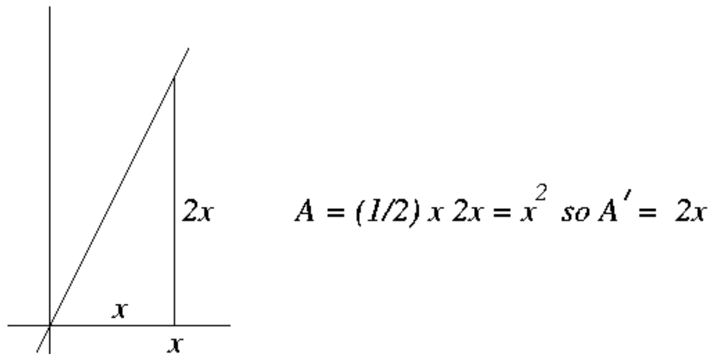
3. $f''(x) = \sin(x)$, $f(\pi/2) = -1$, and $f'(\pi/2) = 2$.

   $f'(x) = -\cos(x)+C$, giving $2 = f'(\pi/2) = -0+C$, so $C = 2$ and $f'(x) = -\cos(x)+2$. We then find that $f(x) = -\sin(x) + 2x + C_1$, and using the condition on $f(\pi/2)$, we have $-1 = f(\pi/2) = -1 + 2 \cdot \frac{\pi}{2} + C_1$, giving $C_1 = -\pi$, so $f(x) = \sin(x) + 2x - \pi$.

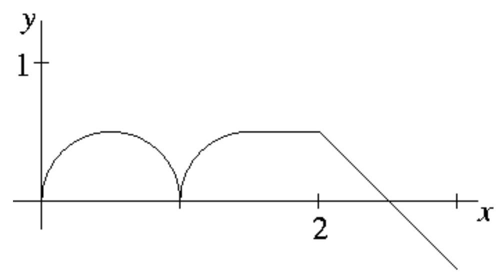
4. $f''(x) = \sin(x)$ and $f(\pi/2) = 3$.

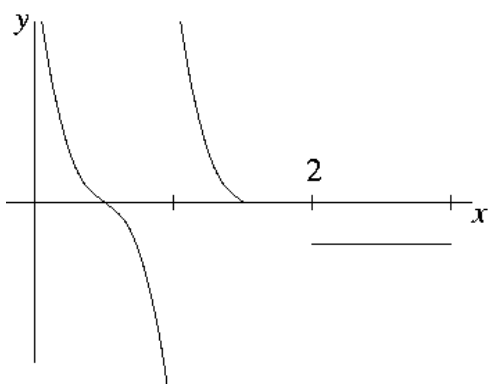
   From part 1. we have $f(x) = -\sin(x) + Cx + C_1$, where C and C_1 are arbitrary constants. Using the condition on $f(\pi/2)$, we have $3 = f(\pi/2) = -1 + C \cdot \frac{\pi}{2} + C_1$, giving $C_1 = 4 - C \cdot \frac{\pi}{2}$, so $f(x) = \sin(x) + C(x - \frac{\pi}{2}) + 4$, where C is an arbitrary constant.

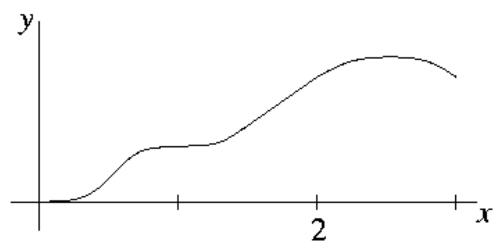
VIII. Let $f(x) = 5x$ for $x \geq 0$. Use the graph of $f(x)$ to determine explicitly the area function $A(x)$ for $f(x)$
(5) (starting at 0). Verify by computation that $A'(x) = f(x)$.



IX. The graph of a certain function $y = f(x)$ is shown at the right. On
(10) two separate graphs, sketch the graph of $f'(x)$, and of a function $F(x)$ for which $F'(x) = f(x)$ and $F(0) = 0$.







X. Verify that if f is even and g is odd, then $f \circ g$ is even. Verify that if f and g are both odd, then $f \circ g$ is
(6) odd.

For f even and g odd, we have $(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = (f \circ g)(x)$, so $f \circ g$ is even.

For f odd and g odd, we have $(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = -f(g(x)) = -(f \circ g)(x)$, so $f \circ g$ is odd.

XI. Calculate each of the following.
(12)

1. $\dfrac{dw}{dz}$ if $\csc(w \cot(z)) = w^3$

$-\csc(w\cot(z))\cot(w\cot(z))\left(-w\csc^2(z) + \frac{dw}{dz}\cot(z)\right) = 3w^2\frac{dw}{dz}$, so

$w\csc^2(z)\csc(w\cot(z))\cot(w\cot(z)) - \frac{dw}{dz}\cot(z)\csc(w\cot(z))\cot(w\cot(z)) = 3w^2\frac{dw}{dz}$

and therefore $\dfrac{dw}{dz} = \dfrac{w\csc^2(z)\csc(w\cot(z))\cot(w\cot(z))}{\cot(z)\csc(w\cot(z))\cot(w\cot(z)) + 3w^2}$.

2. $G'(x)$, if $G(x) = L(1/L(x))$ and $L'(x) = 1/x$

$$G'(x) = L'(1/L(x)) \cdot (1/L(x))' = (1/(1/L(x))) \cdot (-1)(L(x))^{-2}L'(x) = -\frac{L(x)}{(L(x)^2)} \cdot \frac{1}{x} = -\frac{1}{xL(x)}.$$

3. the derivative of $f(g^2(x))g(f^2(x))$

$$\begin{aligned} &f'(g^2(x))(g^2(x))'g(f^2(x)) + f(g^2(x))g'(f^2(x))(f^2(x))' \\ &= 2f'(g^2(x))g(x)g'(x)g(f^2(x)) + 2f(g^2(x))g'(f^2(x))f(x)f'(x)\ . \end{aligned}$$

XII. (6) Write a precise definition of $\lim_{x\to a} f(x) = L$. Write a precise definition of $\lim_{x\to -\infty} f(x) = -\infty$.

$\lim_{x\to a} f(x) = L$ means that for every positive number ϵ, there exists a positive number δ so that if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.
$\lim_{x\to -\infty} f(x) = -\infty$ means that for every number M, there exists a number N so that if $x < N$, then $f(x) < M$.

XIII. (8) Recall that the rate of change of a function $f(x)$ at the x-value a is the unique number $f'(a)$ for which $f(a+h) = f(a) + f'(a)h + \epsilon(h)$ with $\lim_{h\to 0} \frac{\epsilon(h)}{h} = 0$ (if such a number m exists). Assuming that the rate of change of f does exist at the x-value a, find the rate of change of the function f^2 at a by writing $f(a+h)$ as $f(a) + f'(a)h + \epsilon(h)$, squaring this, and applying this description of the rate of change.

$$\begin{aligned} f^2(a+h) &= (f(a) + f'(a)h + \epsilon(h))^2 \\ &= f^2(a) + 2f(a)f'(a)h + (f'(a))^2h^2 + 2f(a)\epsilon(h) + \epsilon^2(h) + 2f'(a)\epsilon(h)h\ . \end{aligned}$$

Putting $\epsilon_{f^2}(h) = (f'(a))^2h^2 + 2f(a)\epsilon(h) + \epsilon^2(h) + 2f'(a)\epsilon(h)h$, we calculate that

$$\begin{aligned} \lim_{h\to 0} \frac{\epsilon_{f^2}(h)}{h} &= \lim_{h\to 0} (f'(a))^2h + 2f(a)\frac{\epsilon(h)}{h} + \epsilon(h)\frac{\epsilon(h)}{h} + 2f'(a)\epsilon(h) \\ &= (f'(a))^2 \cdot 0 + 2f(a) \cdot 0 + 0 \cdot 0 + 2f'(a) \cdot 0 = 0\ , \end{aligned}$$

so the rate of change of f^2 at $x = a$ is $2f(a)f'(a)$.

XIV. (4) State the Intermediate Value Theorem.

Suppose that $f(x)$ is a continuous function for $a \le x \le b$. If N is any number between $f(a)$ and $f(b)$, then there exists a number c with $a < c < b$ such that $f(c) = N$.

XV. (6) Solve the following related rates problem: A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

We are given that $\frac{d\theta}{dt} = 8\pi$ radians/min (since 1 revolution is 2π radians), and we want $\left.\frac{ds}{dt}\right|_{x=4}$. We have $\tan(\theta) = \frac{s}{3}$. Taking the derivative with respect to t, we have $\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{3}\frac{ds}{dt}$. Specializing to the moment when $s = 1$, when $\sqrt{s^2+9} = \sqrt{10}$, we find that $\frac{ds}{dt} = 3\left(\frac{\sqrt{10}}{3}\right)^2 8\pi = \frac{80\pi}{3}$ km/min.