Math 6833 homework problems

19. Let $\pi\colon E \to B$ be a continuous map. A *local cross-section* at b is a map $s\colon U \to E$, where U is an open neighborhood of b, such that $\pi \circ s$ is the identity on U, and one says that π *has local cross sections* if it has a local cross section at each point of B. Let $\pi\colon \mathcal{T}_S \to \mathbb{R}^m_{>0}$ send h to $(L_{\alpha_1}(h), \ldots, L_{\alpha_m}(h))$ as discussed in class. Prove that π has local cross-sections. Remark: the local product structure $h\colon U \times \mathbb{R}^m \to \pi^{-1}(U)$ of $\mathcal{T}_S$ is then defined by $h(u, y) = y \cdot s(u)$.

20. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, a linear transformation of the plane $\mathbb{R}^2$. Find its eigenvalues λ and $1/\lambda$, where $\lambda > 1$, and find a pair of eigenvectors $\{v_1, v_2\}$, for which v_1 has eigenvalue λ and v_2 has eigenvalue $1/\lambda$. Let e_1, e_2 be the standard basis, and graph the integer lattice $\mathbb{Z}e_1 \times \mathbb{Z}e_2$ in the v_1, v_2 basis.

21. Let f be the linear transformation of $\mathbb{R}^2$ which is multiplication by the matrix A in the previous problem, let μ_s and μ_u be the measures associated to the stable and unstable foliations associated to A. Explain how the push-forward $f\,\mu_u$ equals $(1/\lambda)\,\mu_u$.

22. Recall that $\mathrm{diff}(S)$ is the connected component of the identity in $\mathrm{Diff}(S)$. Observe that if $g \in \mathrm{Diff}_+(S)$ and h is a Riemannian metric on S, then the push forward $g\,h$ equals h if and only if g is an isometry of h.

    1. Our first definition of $\mathcal{T}_S$ was the equivalence classes of hyperbolic metrics on S, where $h_1 \sim h_2$ when there exists $j \in \mathrm{diff}(S)$ such that $j\,h_1 = h_2$. For this definition, the action of $\mathcal{H}_+(S)$ on $\mathcal{T}_S$ is $\langle g\rangle\,[h] = [g\,h]$. Using this definition, prove that $\langle g\rangle\,[h] = [h]$ if and only if g is isotopic to an isometry of S when S has the metric h.
    2. Our second definition of $\mathcal{T}_S$ was the equivalence classes of marked hyperbolic structures on S, that is, pairs (S_1, g_1) with S_1 a surface with a hyperbolic metric h_1 and $g_1\colon S_1 \to S$ is a diffeomorphism, with $(S_1, g_1) \sim (S_2, g_2)$ when $g_2^{-1}g_1$ is isotopic to an isometry. For this definition, the action of $\mathcal{H}_+(S)$ on $\mathcal{T}_S$ is $\langle g\rangle\,[(S_1, g_1)] = [(S_1, gg_1)]$. Using this definition, prove that $\langle g\rangle\,[(S_1, g_1)] = [(S_1, g_1)]$ if and only if g is isotopic to an isometry of S, where S has the push-forward metric g_1h_1.