Math 6833 homework problems

14. Suppose that a group G acts on a set X. Recall that if $x \in X$, then the *stabilizer* of x is $G_x = \{g \in G \mid gx = x\}$.
    1. Show that if $x, y \in X$ and x and y lie in the same orbit, then G_x and G_y are conjugate subgroups of X.
    2. Give an example for which the converse is false (find an example that is an effective action, that is, an action where the only element of G that acts as the identity on X is the identity element of G).

15. Let T be the torus and $X = C(T)$ with the action of $\mathcal{H}(T) \cong \mathrm{GL}(2, \mathbb{Z})$. Find the stabilizer of the point $[M]$. Describe its group structure.

16. Consider the isometry $\gamma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ of the upper half-plane $\mathbb{H}^2$.
    1. Calculate the two fixed points of γ on $\mathbb{R}$. They are the endpoints of the axis (i. e. the invariant geodesic) of γ. Calculate the point p where this axis meets the y-axis. Calculate $\gamma(p)$.
    2. Describe geometrically the action of γ on $\mathbb{H}^2$, and draw a fundamental domain $\mathcal{F}$ for the action.
    3. Parameterize the portion of the axis of γ between $\gamma(p)$ and p, and use the hyperbolic metric to carry out a direct calculation that its length is $\ln\left(\frac{7+3\sqrt{5}}{2}\right)$. Notice that this is the length of the unique closed geodesic in the annulus $\mathbb{H}/\gamma$.

17. Consider the isometry $\gamma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ of the upper half-plane $\mathbb{H}^2$. The trace of γ is 3, so its characteristic polynomial is $\lambda^2 - 3\lambda + 1$. The roots $\frac{3 \pm \sqrt{5}}{2}$ of the characteristic polynomial are the eigenvalues of γ, so γ is conjugate in $\mathrm{SL}(2, \mathbb{R})$ to the isometry $\gamma_1 = \begin{pmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2} \end{pmatrix}$. Consequently, the annulus $\mathbb{H}^2/\gamma$ is isometric to the annulus $\mathbb{H}^2/\gamma_1$. Find the axis of γ_1 and a fundamental domain for the action of γ_1. Use a fact from class to see quickly that the length of the unique closed geodesic in $\mathbb{H}^2/\gamma_1$ is $\ln\left(\frac{7+3\sqrt{5}}{2}\right)$.

18. For an ideal quadrilateral in the hyperbolic plane, let d_1 and d_2 be the minimum distances between the opposite pairs of sides, chosen so that $d_1 \geq d_2$. Prove that the isometry classes of ideal quadrilaterals correspond to the interval $[1, \infty)$, with the correspondence given by sending the isometry class to the ratio d_1/d_2. Hint: