Math 6833 homework problems

8. For the annulus $A$, show that $H(A) \cong C_2 \times C_2$. (Ingredients: the homomorphism $H(A) \to \text{Out}(\pi_1(A))$, the homomorphism $H(A) \to C_2$ defined by the way a homeomorphism permutes the components of $\partial A$, and the fact that every element of $H(A \text{ rel } \partial A)$ can be represented by one of the $h_n$.)

9. Let $j : S^1 \to S^1 \times S^1$ be an imbedding. Let $C = S^1 \times \{1\} \subset S^1 \times S^1$, and suppose that $j$ is homotopic to an imbedding that carries $S^1$ to $C$.

1. An imbedding of a manifold $X$ into a manifold $Y$ is called proper if the preimage of $\partial Y$ equals $\partial X$. Suppose that $\beta$ is a properly-imbedded arc in $S^1 \times I$, whose endpoints lie in $S^1 \times \{0\}$, so that $\beta$ cuts $S^1 \times \{0\}$ into two arcs, $\beta_1$ and $\beta_2$. Prove that for either $i = 1$ or $i = 2$, $\beta$ and $\beta_i$ bound a disk in $S^1 \times I$. Note: if $S_\beta$ is the result of cutting a surface $S$ along a properly-imbedded arc $\beta$, then $\chi(S_\beta) = \chi(S) + 1$ (why?).

2. Suppose that $\alpha : S^1 \to S^1 \times I$ is an imbedding whose image is disjoint from $S^1 \times \partial I$. Show that $\alpha(S^1)$ is either contractible, or is parallel to both $S^1 \times \{0\}$ and $S^1 \times \{1\}$.

3. Let $p : S^1 \times \mathbb{R} \to S^1 \times S^1$ be $p(x,r) = (x,r)$, where each $S^1$ is regarded as $\mathbb{R}/\mathbb{Z}$. Explain why there is a lift $\tilde{j} : S^1 \to S^1 \times \mathbb{R}$ such that $j = p\tilde{j}$.

4. Using an argument similar to the one used in the proof that $H(A \text{ rel } \partial A) \cong \mathbb{Z}$, show that $j$ is isotopic to an imbedding that carries $S^1$ to $C$.

10. Check that if $j_0 : S^1 \times I \to S$ and $j_1 : S^1 \times I \to S$ are isotopic imbeddings, then the Dehn twists $T_0$ and $T_1$ that they define are isotopic. You may assume a version of the Isotopy Extension Theorem that tells you that if $j_t$ is an isotopy of imbeddings from $j_0$ to $j_1$, then there is an isotopy of diffeomorphisms $J_t : S \to S$ such that $J_0$ is the identity map and $J_t \circ j_0 = j_t$ for all $t$. Then define $T_t$ using $J_t$ and $T_0$.

11. Let $C$ be a contractible loop in $S$. Prove that $t_C$ is isotopic to the identity. Hint: $C$ bounds a disk. We may use any imbedding of $S^1 \times I$ to define $t_C$, choose a nice one.

12. Let $f : S \to S$ be an orientation-reversing diffeomorphism, and let $C$ be a loop in $S$. Draw convincing pictures showing that $f t_C f^{-1} = t_{f(C)}^{-1}$.

13. For each $k = 0, 1, 2, 3$, find a pair of 4-simplices of $C(F(3,0))$ whose intersection is a $k$-simplex.