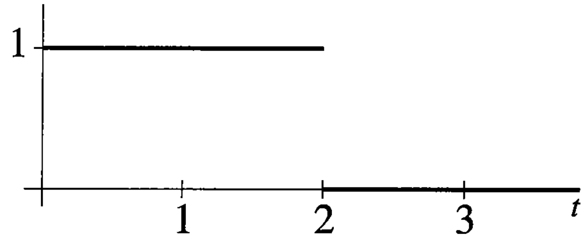
Mathematics 3113-001 Name (please print) ______________________

Examination III Form B

November 20, 2003

I. The figure to the right shows the graph of a function $f(t)$. Using the *definition* of the Laplace transform to calculate $\mathcal{L}\{f(t)\}$.
(4)



$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st} \cdot 1\,dt + \int_2^\infty e^{-st} \cdot 0\,dt$$

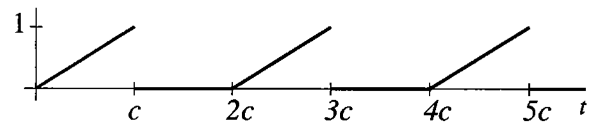
$$= -\frac{1}{s} e^{-st}\Big|_0^2 + 0 = \frac{1}{s} - \frac{e^{-2s}}{s}$$

II. For the function $f(t)$ shown in the previous problem, write an expression for $f(t)$ using the Heaviside step function $u_2(t)$, and use this expression and formulas from your table of Laplace transforms to calculate the Laplace transform of $f(t)$.
(4)

$$f(t) = 1 - u_2(t)$$

$$\mathcal{L}(f(t)) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

III. Calculate the Laplace transform of this periodic function:
(6)



period $= 2c$

$$\mathcal{L}(f(t)) = \frac{1}{1-e^{-2cs}} \int_0^{2c} e^{-st} f(t)\,dt.$$

$$\int_0^{2c} e^{-st} f(t)\,dt = \int_0^c e^{-st} \cdot \frac{t}{c}\,dt + \int_c^{2c} e^{-st} \cdot 0\,dt$$

$$\boxed{\begin{array}{ll} u = \frac{t}{c} & v = -\frac{1}{s}e^{-st} \\ du = \frac{1}{c}dt & dv = e^{-st} \end{array}}$$

$$= -\frac{t}{cs} e^{-st}\Big|_0^c + \frac{1}{cs}\int_0^c e^{-st}\,dt + 0$$

$$= -\frac{1}{s} e^{-cs} + -\frac{1}{cs^2} e^{-st}\Big|_0^c = -\frac{1}{s} e^{-cs} - \frac{1}{cs^2} e^{-cs} + \frac{1}{cs^2}$$

Therefore $\mathcal{L}(f(t)) = \dfrac{1}{1-e^{-2cs}} \left(-\frac{1}{s} e^{-cs} - \frac{1}{cs^2} e^{-cs} + \frac{1}{cs^2} \right)$

IV. (16) Calculate the inverse Laplace transforms of the following functions.

1. $\dfrac{1}{(s-4)^5}$

$$\mathcal{L}^{-1}\left(\frac{1}{s^5}\right) = \frac{t^4}{4!}$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s-4)^5}\right) = \frac{e^{5t}\,t^4}{4!}$$

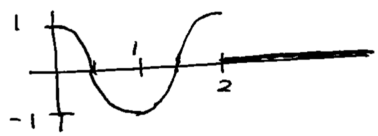
2. $\dfrac{s(1-e^{-2s})}{s^2+\pi^2}$ (graph the resulting $f(t)$)

$$= \frac{s}{s^2+\pi^2} - e^{-2s}\frac{s}{s^2+\pi^2}$$

$$f(t) = \cos(\pi t) - u_2(t)\cos(\pi(t-2))$$

$$= \cos(\pi t) - u_2(t)\cos(\pi t - 2\pi) = \cos(\pi t) - u_2(t)\cos(\pi t)$$

period of $\cos(\pi t) = \frac{2\pi}{\pi} = 2$, so graph is



3. $\dfrac{s^2}{(s-1)(s-2)(s-3)} = \dfrac{A}{s-1} + \dfrac{B}{s-2} + \dfrac{C}{s-3}$

$$s^2 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

For $s=1$, $1 = 2A$, $A = \frac{1}{2}$

For $s=2$, $4 = -B$, $B = -4$

For $s=3$, $9 = 2C$, $C = \frac{9}{2}$

$$\mathcal{L}^{-1}\left(\frac{1}{2}\frac{1}{s-1} - 4\frac{1}{s-2} + \frac{9}{2}\frac{1}{s-3}\right) = \frac{1}{2}e^{t} - 4e^{2t} + \frac{9}{2}e^{3t}$$

4. $\ln\left(1+\dfrac{1}{s^2}\right)$ (hint: it is best to simplify first, using properties of the logarithm function)

$$= \ln\left(\frac{s^2+1}{s^2}\right) = \ln(s^2+1) - \ln(s^2)$$

$$\mathcal{L}(-t\,f(t)) = \frac{d}{ds}\left(\ln(s^2+1) - \ln(s^2)\right) = \frac{2s}{s^2+1} - \frac{2}{s}$$

$$-t\,f(t) = 2\cos(t) - 2$$

$$f(t) = \frac{2 - 2\cos(t)}{t}$$

V. (8) Use Laplace transforms to solve the following system of differential equations. Carry out the inverse transform without using partial fractions.

$y' = x + y,\ x' = 3y - x,\ x(0) = 0,\ y(0) = 1$

$sY - 1 = X + Y$

$sX = 3Y - X$

$-X + (s-1)Y = 1$

$(s+1)X - 3Y = 0$

$$\begin{vmatrix} -1 & s-1 \\ s+1 & -3 \end{vmatrix} X = \begin{vmatrix} 1 & s-1 \\ 0 & -3 \end{vmatrix}$$

$$(3 - (s^2-1))X = -3$$

$$X = \frac{3}{s^2-4} = \frac{3}{2}\,\frac{2}{s^2-4}, \quad \text{so} \quad x(t) = \frac{3}{2}\sinh(2t)$$

$$\begin{vmatrix} -1 & s-1 \\ s+1 & -3 \end{vmatrix} Y = \begin{vmatrix} -1 & 1 \\ s+1 & 0 \end{vmatrix}$$

$$(4-s^2)Y = -s-1$$

$$Y = \frac{s+1}{s^2-4} = \frac{s}{s^2-4} + \frac{1}{2}\,\frac{2}{s^2-4}$$

$$y(t) = \cosh(2t) + \frac{1}{2}\sinh(2t)$$

VI. (6) Use Laplace transforms to solve the following initial value problem. Express the answer using an integral involving $f(t)$: $x'' + x = f(t),\ x(0) = x'(0) = 0$.

$$s^2X + X = \mathcal{L}(f(t))$$

$$X = \frac{1}{s^2+1}\cdot\mathcal{L}(f(t)) = \mathcal{L}(\sin(t))\,\mathcal{L}(f(t))$$

$$x(t) = \sin(t) * f(t) = \int_0^t \sin(\sigma)\, f(t-\sigma)\,d\sigma$$

VII. Use Laplace transforms to solve the following initial value problem, and **graph the solution:**
(6) $x^{(3)} = 2\,\delta(t-1),\ \ x(0) = x'(0) = x''(0) = 0$

$$s^3 X = 2e^{-s}$$

$$X = e^{-s}\cdot\frac{2}{s^3} = e^{-s}\,\mathcal{L}(t^2)$$

$$x(t) = u_1(t)\,(t-1)^2$$

VIII. Use the definition of convolution to calculate $(t-1) * t$.
(4)

$$(t-1)*t = \int_0^t (\sigma-1)(t-\sigma)\,d\sigma$$

$$= \int_0^t -\sigma^2 + t\sigma + \sigma - t\,d\sigma$$

$$= -\frac{\sigma^3}{3} + \frac{t\sigma^2}{2} + \frac{\sigma^2}{2} - t\sigma\,\Big|_0^t$$

$$= -\frac{t^3}{3} + \frac{t^3}{2} + \frac{t^2}{2} - t^2$$

$$= \frac{t^3}{6} - \frac{t^2}{2}$$

IX. Write a function $f(t)$ whose derivative is e^{-t^2} and for which $f(0) = \sqrt{11}$.
(3)

By the Fundamental Theorem of Calculus, $f(t) = \sqrt{11} + \int_0^t e^{-\sigma^2}\,d\sigma$