Mathematics 3113-001 Name (please print) ____________________

Examination II Form A

October 23, 2003

I. (8) Using the formula $x^s\big((A_0 + A_1x + \cdots + A_mx^m)e^{rx}\cos(kx) + (B_0 + B_1x + \cdots + B_mx^m)e^{rx}\sin(kx)\big)$, write trial solutions for the method of undetermined coefficients for the following differential equations, but *do not* substitute them into the equations or proceed further with finding the solution.

1. $y'' + y' = e^x\cos(3x)$ $\quad y''+y'=0,\quad r^2+r=0,\quad r(r+1)=0,\quad r=0,-1,$

$y_c = c_1 + c_2e^{-x}$

For $e^x\cos(3x)$, $y_p = x^s\left[Ae^x\cos(3x) + Be^x\sin(3x)\right]$

No duplicate terms, so take $s=0$.

$y_p = Ae^x\cos(3x) + Be^x\sin(3x)$

2. $y'' + y' = x^2e^{-x}$

For x^2e^{-x}, $\quad y_p = x^s\left[\underbrace{(A_0 + A_1x + A_2x^2)e^{-x}}\right]$

one duplicate term, A_0e^{-x}, so need $s=1$

$$y_p = (A_0x + A_1x^2 + A_2x^3)e^{-x}$$

II. (8) Given that $r^{12} + 3r^{11} + 6r^{10} + 9r^9 + 10r^8 + 9r^7 + 6r^6 + 3r^5 + r^4 = r^4(r+1)(r^2+r+1)(r^2+1)^2(r+1)$, write the general solution to $y^{(12)} + 3y^{(11)} + 6y^{(10)} + 9y^{(9)} + 10y^{(8)} + 9y^{(7)} + 6y^{(6)} + 3y^{(5)} + y^{(4)} = 0$.

$r^2+r+1=0,\quad r = (-1 \pm \sqrt{1-4})/2 = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$

Roots are $0,0,0,0,-1,-1,\ -\frac{1}{2}\pm\frac{\sqrt{3}}{2},\ \pm i,\ \pm i$

$$\begin{aligned} y = {} & c_1 + c_2x + c_3x^2 + c_4x^3 \\ & + c_5e^{-x} + c_6xe^{-x} \\ & + c_7e^{-\frac{1}{2}x}\cos\left(\tfrac{\sqrt{3}}{2}x\right) + c_8e^{-\frac{1}{2}x}\sin\left(\tfrac{\sqrt{3}}{2}x\right) \\ & + c_9\cos x + c_{10}\sin x + c_{11}x\cos x + c_{12}x\sin x \end{aligned}$$

III. (5) Transform the differential equation $t^3x^{(3)} - 2t^2x'' + 3tx' + 5x = \ln(t)$ into an equivalent system of first-order differential equations.

$x_1 = x$

$x_2 = x'$

$x_3 = x''$

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ t^3x_3' - 2t^2x_3 + 3t\,x_2 + 5x_1 &= \ln(t) \end{aligned}}$$

IV. (10) Use the method of variation of parameters to find a particular solution of the differential equation $y''+y'=x$ as follows.

1. Given that $y_1 = 1$ and $y_2 = e^{-x}$ are two linearly independent solutions of the associated homogeneous equation $y''+y'=0$, write the general equations $y_1u_1' + y_2u_2' = 0$, $y_1'u_1' + y_2'u_2' = f(x)$ of the method of variation of parameters to find u_1' and u_2' for the nonhomogeneous linear equation $y''+y'=x$.

$$1 \cdot u_1' + e^{-x} u_2' = 0$$
$$0 \cdot u_1' - e^{-x} u_2' = x$$

2. Solve for u_1' and find u_1.

Adding the equations gives $u_1' = x$

$$u_1 = \frac{x^2}{2}$$

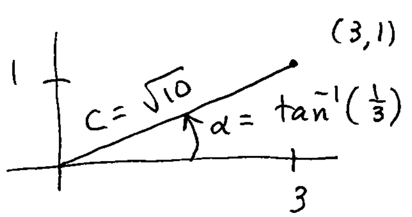
3. Solve for u_2'. Use the integration formula $\int xe^x\,dx = (x-1)e^x$ to find u_2.

$$-e^{-x} u_2' = x$$
$$u_2' = -xe^{x}$$
$$u_2 = -\int xe^x\,dx = (1-x)e^{x}$$

4. Write the particular solution y_p that has been found.

$$y_p = u_1y_1 + u_2y_2 = \frac{x^2}{2}\cdot 1 + (1-x)e^{x}\cdot e^{-x} = \frac{x^2}{2} - x + 1$$

V. (5) Write $3\cos(3t) + \sin(3t)$ in phase-angle form, leaving an expression involving the inverse tangent function $\tan^{-1}$ in the answer (i. e. do not perform a numerical approximation).



$$\sqrt{10}\,\cos\left(3t - \tan^{-1}\left(\tfrac{1}{3}\right)\right)$$

VI. (6) Use the method of elimination to solve the linear system. Use differential operators, if you prefer, or else just solve the second equation for x and use that expression in the first equation.

$$x' = x - 2y,$$
$$y' = 2x - 3y$$

$y' = 2x - 3y$, $x = \frac{y'}{2} + \frac{3}{2}y$, $x' = \frac{y''}{2} + \frac{3}{2}y'$

$\frac{y''}{2} + \frac{3}{2}y' = \frac{y'}{2} + \frac{3}{2}y - 2y$

$\frac{y''}{2} + y' + \frac{y}{2} = 0$, $y'' + 2y' + y = 0$, $r^2 + 2r + 1 = 0$, $r = -1, -1$

$y = c_1 e^{-t} + c_2 t e^{-t}$

$y' = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t}$

$x = \frac{y'}{2} + \frac{3}{2}y = -\frac{c_1}{2} e^{-t} + \frac{c_2}{2} e^{-t} - \frac{c_2}{2} e^{-t} + \frac{3c_1}{2} e^{-t} + \frac{3c_2}{2} t e^{-t}$

$x = \left(c_1 + \frac{c_2}{2}\right) e^{-t} + c_2 t e^{-t}$

$y = c_1 e^{-t} + c_2 t e^{-t}$

VII. (4) Consider the boundary-value problem $y'' + \lambda y = 0$, $y(a) = a_0$, $y(b) = b_0$.

1. Define what it means to say that a value of λ *is an eigenvalue* of this problem.

   It means that for this value of λ, the problem has a nonzero solution.

2. Define what it means to say that a function y *is an eigenfunction* associated to an eigenvalue λ.

   It means that y is one of the nonzero solutions of the problem when λ has that value.

VIII. Consider the boundary-value problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(\pi) = 0$.
(10)

1. Show that $\lambda = 0$ is not an eigenvalue of this problem.

$y'' = 0$, $\quad y = c_1 + c_2 x$

$0 = y(0) = c_1$ so $y = c_2 x$

$0 = y(\pi) = c_2 \pi$ so $c_2 = 0$

$\therefore y = 0$ is the only solution

$\therefore \lambda = 0$ is not an eigenvalue

2. By writing $\lambda = \alpha^2$, $\alpha > 0$, find all positive eigenvalues, and an associated eigenfunction for each positive eigenvalue.

$y'' + \alpha^2 y = 0$

$y = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$

$0 = y(0) = c_1 \cdot 1 + c_2 \cdot 0 = c_1$, so $y = c_2 \sin(\alpha x)$

$0 = y(\pi) = c_2 \sin(\alpha \pi)$. If $y \neq 0$, then $c_2 \neq 0$, so

$0 = \sin(\alpha \pi)$

$\therefore \alpha \pi = n\pi$ for some $n = 1, 2, 3, \ldots$

$\therefore \alpha = n$ for some $n = 1, 2, 3, .$

and $y = c_2 \sin(nx)$

eigenvalues	eigen functions	
n^2	$\sin(nx)$	$n = 1, 2, 3, \ldots$

X. (Bonus problem) Simplify $(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})^{100}$.
(4)

$\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)^{100} = \left(\cos(\frac{\pi}{4}) + i \sin(\frac{\pi}{4})\right)^{100} = \left(e^{i\frac{\pi}{4}}\right)^{100}$

$= e^{25\pi i} = \cos(25\pi) + i \sin(25\pi)$

$= \cos(\pi) + i \sin(\pi) = -1 + i \cdot 0 = -1$