Mathematics 3113-001 Name (please print) ______________

Examination I Form B

September 23, 2003

I. (5) Use a substitution of the form $v = y^{1-r}$ to transform the equation $y^3 y' + \sin(x) = 3y^4 x$ into a linear DE. Find an *integrating factor* for the linear DE, but *do not* proceed further with solving the linear DE or the original DE.

$$y' - 3xy = -\sin(x)\, y^{-3}, \quad r = -3, \quad v = y^4, \quad v' = 4y^3 y'$$

$$\frac{1}{4y^3} v' - 3xy = -\sin(x)\, y^{-3}$$

$$v' - 12x\, y^4 = -4\sin x$$

$$v' - 12x\, v = -4\sin x$$

$$\int -12x\, dx = -6x^2$$

integrating factor is e^{-6x^2}

II. (6) A 1000-liter tank initially contains 200 liters of brine containing 20 kilograms of salt. Brine containing 0.2 kilogram of salt per liter enters the tank at the rate of 5 liters/sec, and the perfectly mixed brine in the tank flows out at a rate of 2 liters/sec.

1. Let $V(t)$ be the volume of brine in the tank as a function of time. Write an explicit expression for $V(t)$.

$$V = 200 + 3t$$

2. Let $x(t)$ be the amount of salt in the tank in kilograms. Since the concentration of salt in the tank is $x(t)/V(t)$, the rate of change x' is related to x by the linear differential equation $x' = 1 - 2x/V$. Write an initial value problem whose solution is x, but *do not* solve it.

$$x' = 1 - 2x/V$$

$$x' + \frac{2}{200+3t}\, x = 1, \quad x(0) = 20$$

3. Given that the solution to the initial value problem in part 2 is $x(t) = (200+3t)/5 + 160\,(200)^{2/3}\,(200+3t)^{-2/3}$, calculate the exact amount (not a decimal approximation) of salt in the tank at the moment when it contains 800 liters of brine. Your answer should be simplified, but since it is to be exact, it will involve a fractional exponent.

$$V = 200 + 3t = 800, \quad t = 200$$

$$x(200) = \frac{800}{5} + 160 \cdot \frac{(200)^{2/3}}{(800)^{2/3}} = 160 + \frac{160}{4^{2/3}}$$

$$\left(\text{which equals } 160 + \frac{160}{16^{1/3}} = 160 + \frac{160}{2^{1/3}\, 8^{1/3}} = 160 + \frac{80}{2^{1/3}}\right)$$

III. (9) This problem concerns the initial value problem $y' = 4y^{3/4},\ \ y(1) = 0$.

1. Divide through by $y^{3/4}$ and use the method of separation of variables to find a general solution to the DE $y' = 4y^{3/4}$.

$$y^{-3/4}\,dy = 4\,dx$$
$$4y^{1/4} = 4x + C_1$$
$$y^{1/4} = x + C$$
$$y = (x + C)^4$$

2. Use the solution you have found to solve the initial value problem $y' = 4y^{3/4},\ \ y(1) = 0$.

$$0 = y(1) = (1 + c)^4, \quad c = -1$$
$$y = (x - 1)^4$$

3. Find a different solution to the initial value problem $y' = 4y^{3/4},\ \ y(1) = 0$.

$$y = 0$$

4. If you can, find a third solution to the initial value problem $y' = 4y^{3/4},\ \ y(1) = 0$.

$$y(x) = \begin{cases} 0 & x \leq 1 \\ (x-1)^4 & x \geq 1 \end{cases}$$

IV. (4) Use the characteristic equation to find a general solution of $2y' = 2y'' + \frac{1}{2}y$.

$$2y'' - 2y' + \tfrac{1}{2}y = 0$$
$$2r^2 - 2r + \tfrac{1}{2} = 0$$
$$4r^2 - 4r + 1 = 0$$
$$r = \tfrac{1}{2}, \tfrac{1}{2}$$
$$y = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}$$

V. (3) On what interval does the Existence and Uniqueness Theorem for linear DE's guarantee that the initial value problem $x^2y'' + y = 0$, $y(2) = 3$, $y'(2) = 16$ has a unique solution?

$y'' + \frac{1}{x^2}y = 0$, $\frac{1}{x^2}$ continuous at all $x \neq 0$. The largest interval containing 2 on which $\frac{1}{x^2}$ is continuous is

$I = (0, \infty)$

VI. (8) It is known that $\sin(2x)$ and $\cos(2x)$ are two linearly independent solutions to the second-order linear DE $y'' + 4y = 0$.

1. Solve the initial value problem $y'' + 4y = 0$, $y(0) = 10$, $y'(0) = -5$.

$y = c_1 \sin(2x) + c_2 \cos(2x)$
$y' = 2c_1 \cos(2x) - 2c_2 \sin(2x)$
$10 = y(0) = c_2$
$-5 = y'(0) = 2c_1$
$y = -\frac{5}{2}\sin(2x) + 10\cos(2x)$

2. Given that $y = e^{-x}$ is a solution to $y'' + 4y = 5e^{-x}$, solve the initial value problem $y'' + 4y = 5e^{-x}$, $y(0) = -5$, $y'(0) = 10$.

$y = c_1 \sin(2x) + c_2\cos(2x) + e^{-x}$
$y' = 2c_1\cos(2x) - 2c_2 \sin(2x) - e^{-x}$
$-5 = y(0) = c_2 + 1$, $c_2 = -6$
$10 = y'(0) = 2c_1 - 1$, $c_1 = \frac{11}{2}$
$y = \frac{11}{2}\sin(2x) - 6\cos(2x) + e^{-x}$

VII. (6) Let $f_1, f_2, \ldots, f_r$ be functions defined on an interval I.

1. Define what it means to say that $f_1, f_2, \ldots, f_r$ are *linearly dependent* on I.

There exist constants $k_1, k_2, \ldots, k_r$, not all 0, such that $k_1f_1 + k_2f_2 + \ldots + k_rf_r = 0$

2. Show, using the definition, that the functions $(x+2)^2$, x^2, and $2x+2$ are linearly dependent on the real line.

$(x+2)^2 = x^2 + 4x + 4 = x^2 + 2(2x+2)$

$1\cdot(x+2)^2 + (-1)\cdot x^2 + (-2)(2x+2) = 0$

VIII. Find a general solution to $2x\,y' + 4x^2 = y$.
(4)

$$y' - \frac{1}{2x} y = -2x \quad \text{linear}$$

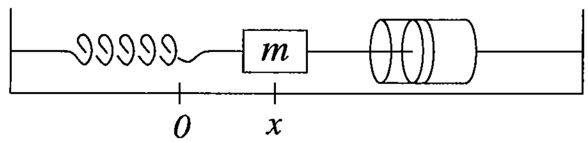
$$\int -\frac{1}{2x}\,dx = -\frac{1}{2}\ln x, \quad e^{-\frac{1}{2}\ln x} = e^{\ln(x^{-\frac{1}{2}})} = x^{-\frac{1}{2}}$$

$$x^{-\frac{1}{2}} y' - \frac{1}{2} x^{-\frac{3}{2}} = -2x^{\frac{1}{2}}$$

$$x^{-\frac{1}{2}} y = \int -2x^{1/2}\,dx = -2 \cdot \frac{2}{3} x^{3/2} + C = -\frac{4}{3} x^{3/2} + C$$

$$y = -\frac{4}{3} x^2 + C x^{\frac{1}{2}}$$

IX. The figure to the right shows a simple spring-dashpot system. The variable x shows the position of the mass, with the value $x = 0$ corresponding to the natural length of the spring. The spring exerts a restoring force of $-kx$, while the dashpot resists the motion with a force proportional to the velocity x'.
(4)



1. Express the acceleration a in terms of x.

$$a = x''$$

2. Use Newton's Law $F = ma$ to write a differential equation relating x to the velocity and the acceleration. Write the differential equation in the standard form for a linear differential equation.

$$mx'' = ma = F = -kx - cx'$$

$$mx'' + cx' + kx = 0$$

X. Use the substitution $v = y' = dy/dx$ to transform the DE $y'' = 3y(y')^2$ into a first-order DE of the form $F(y, v, \frac{dv}{dy}) = 0$, but *do not* proceed further with solving either of the DE's.
(4)

$$v = y'$$

$$y'' = v' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}$$

$$v\frac{dv}{dy} = 3yv^2$$

$$\frac{dv}{dy} - 3vy = 0$$