Mathematics 3113-001 Name (please print) ____________________

Final Examination Form B

December 15, 2003

I. (4) Let $f(t) = t$ for $0 \le t < \pi^3$, and $f(t) = 0$ for $t \ge \pi^3$. Use the *definition* of the Laplace transform to calculate $\mathcal{L}\{f(t)\}$.

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^{\pi^3} te^{-st}\,dt + \int_{\pi^3}^\infty 0 \cdot e^{-st}\,dt$$

$u = t \quad v = -\frac{1}{s}e^{-st}$

$du = dt \quad dv = e^{-st}dt$

$$= -\frac{t}{s}e^{-st}\Big|_0^{\pi^3} + \int_0^{\pi^3} \frac{1}{s}e^{-st}\,dt = -\frac{\pi^3}{s}e^{-\pi^3 s} + -\frac{1}{s^2}e^{-st}\Big|_0^{\pi^3}$$

$$= -\frac{\pi^3}{s}e^{-\pi^3 s} - \frac{1}{s^2}e^{-\pi^3 s} + \frac{1}{s^2}$$

II. (4) Let $f(t) = t$ for $0 \le t < \pi^3$, and $f(t) = 0$ for $t \ge \pi^3$. Use a step function to write an expression for $f(t)$, and use the formulas list for Laplace transforms to calculate $\mathcal{L}\{f(t)\}$.

$$f(t) = t \cdot (1 - u_{\pi^3}(t)) = t - u_{\pi^3}(t)t = t - u_{\pi^3}(t)(t - \pi^3) - \pi^3 u_{\pi^3}(t)$$

$$\mathcal{L}(f(t)) = \frac{1}{s^2} - e^{-\pi^3 s} \cdot \frac{1}{s^2} - \pi^3 \cdot \frac{e^{-\pi^3 s}}{s}$$

III. (4) Solve the initial value problem $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = -1$ by using the characteristic equation to find the general solution, then solving equations involving the initial values to find the solution that satisfies the initial values.

$r^2 + 4 = 0$, $r = \pm 2i$, $y = A\cos(2x) + B\sin(2x)$

$0 = y(0) = A \cdot 1$ so $y = B\sin(2x)$

$y' = 2B\cos(2x)$

$-1 = y'(0) = 2B$ so $B = -\frac{1}{2}$

$$y = -\frac{1}{2}\sin(2x)$$

IV. (4) Solve the initial value problem $x'' + 4x = 0$, $x(0) = 0$, $x'(0) = -1$ by using the Laplace transform.

$s^2 X(s) + 1 + 4X(s) = 0$

$$X(s) = -\frac{1}{s^2+4} = -\frac{1}{2} \cdot \frac{2}{s^2+4}$$

$$x(t) = -\frac{1}{2}\sin(2t)$$

V. (4) Use separation of variables to solve $\dfrac{dy}{dx} = 6e^{2x-y}$.

$$e^y\,dy = 6e^{2x}\,dx$$
$$e^y = 3e^{2x} + C$$
$$y = \ln(3e^{2x} + C)$$

VI. (8) Solve the initial value problem $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = -1$ as follows.

1. Put $y = \displaystyle\sum_{n=0}^{\infty} c_n x^n$, and write an expresssion for y'' as a series. Use these expresssions in the differential equation to find a recursive formula for c_n (it should give c_{n+2} in terms of c_n).

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$$
$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n + 4\sum_{n=0}^{\infty} c_n x^n = 0$$
$$\sum_{n=0}^{\infty} \left((n+2)(n+1)c_{n+2} + 4c_n\right)x^n = 0$$
$$c_{n+2} = -\frac{4}{(n+2)(n+1)}c_n$$

2. Use the initial conditions $y(0) = 0$ and $y'(0) = -1$ to find c_0 and c_1, and solve for c_n, treating separately the cases n even and n odd.

$$c_0 = y(0) = 0, \quad c_1 = y'(0) = -1$$
$$c_2 = -\frac{4}{2\cdot 1}c_0 = 0, \quad c_4 = -\frac{4}{4\cdot 3}c_2 = 0, \quad \text{etc. all } c_{2n} = 0$$
$$c_3 = -\frac{4}{3\cdot 2}c_1, \quad c_5 = -\frac{4}{5\cdot 4}c_3 = \frac{(-4)(-4)}{5\cdot 4\cdot 3\cdot 2}c_1$$
$$c_7 = \frac{(-4)}{7\cdot 6}\cdot\frac{(-4)\cdot(-4)}{5\cdot 4\cdot 3\cdot 2\cdot 1}c_1 = \frac{(-1)^3 4^3}{(2n+1)!}c_1$$
$$c_{2n+1} = \frac{(-1)^n 4^n}{(2n+1)!}c_1 = \frac{(-1)^{n+1}4^n}{(2n+1)!}$$

$$\left(\text{thus } y = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}4^n}{(2n+1)!}x^{2n+1} = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}}{(2n+1)!}x^{2n+1} = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2x)^{2n+1} = -\frac{1}{2}\sin(2x)\right)$$

VII. Solve the initial value problem $x'' + 4x = 0$, $x(0) = 0$, $x'(0) = -1$ as follows.
(6)

1. Putting $x_1 = x$ and $x_2 = x'$, rewrite the equation as a system of two first-order equations. Notice that the initial conditions become $x_1(0) = 0$ and $x_2(0) = -1$.

$$x_1' = x_2$$
$$x_2' + 4x_1 = 0$$

2. Use the Laplace transform and Cramer's rule to find an expression for the Laplace transform $X_1(s)$ of $x_1(t)$, and use the inverse transform to find $x_1(t)$ (of course, there is no need to also find $X_2(s)$ and $x_2(t)$, since $x_1(t)$ is $x(t)$, the solution to the original equation).

$$\begin{aligned} x_1' - x_2 &= 0 \\ 4x_1 + x_2' &= 0 \end{aligned} \xrightarrow{\mathcal{L}} \begin{aligned} sX_1(s) - X_2(s) &= 0 \\ 4X_1(s) + sX_2(s) &= -1 \end{aligned}$$

$$\xrightarrow{C} \begin{vmatrix} s & -1 \\ 4 & s \end{vmatrix} X_1(s) = \begin{vmatrix} 0 & -1 \\ -1 & s \end{vmatrix}$$

$$(s^2+4)X_1(s) = -1$$

$$X_1(s) = -\frac{1}{s^2+4} = -\frac{1}{2}\,\frac{2}{s^2+4}$$

$$x(t) = x_1(t) = -\frac{1}{2}\sin(2t)$$

VIII. Use an integrating factor to solve the first-order linear equation $y' = (1-y)\cos(x)$, $y(\pi) = 2$.
(4)

$$y' + \cos(x)\,y = \cos(x), \qquad \mu = e^{\int \cos(x)\,dx} = e^{\sin x}$$

$$e^{\sin(x)}y' + e^{\sin x}\cdot\cos(x)\,y = e^{\sin(x)}\cdot\cos(x)$$

$$e^{\sin x}\cdot y = \int e^{\sin(x)}\cos(x)\,dx = e^{\sin x} + C$$

$$y = 1 + Ce^{-\sin(x)}$$

$$2 = y(\pi) = 1 + Ce^0, \quad C = 1$$

$$y = 1 + e^{-\sin(x)}$$

IX. Write the Maclaurin series for $\sin(x)$, $\dfrac{\sin(x)}{x}$, $\cosh(x)$, and e^{-x^2}.
(4)

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\frac{\sin(x)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!}(-x^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^{2n} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^4}{4!} - \dots$$

X. If D is the differential operator defined by $Df = f'$, calculate $(D^2 + e^x D + e^{2x})(xe^{3x})$.
(5)

$$(D^2 + e^x D + e^{2x})(xe^{3x}) = D^2(xe^{3x}) + e^x D(xe^{3x}) + e^{2x}\cdot xe^{3x}$$

$$= D(e^{3x} + 3xe^{3x}) + e^x(e^{3x} + 3xe^{3x}) + xe^{5x}$$

$$= 3e^{3x} + 3e^{3x} + 9xe^{3x} + e^{4x} + 3xe^{4x} + xe^{5x}$$

$$= xe^{5x} + (3x+1)e^{4x} + (9x+6)e^{3x}$$

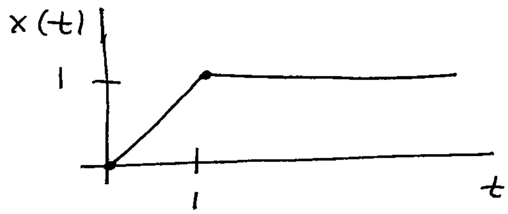
XI. Solve $x'' = \delta(t) - \delta(t-1)$, $x(0) = 0$, $x'(0) = 0$, and graph the solution.
(5)

$$s^2 X(s) = 1 - e^{-s}$$

$$X(s) = \frac{1}{s^2} - e^{-s}\frac{1}{s^2}$$

$$x(t) = t - u_1(t)(t-1)$$

$$= \begin{cases} t & \text{if } t < 1 \\ t - (t-1) = 1 & \text{if } t \geq 1 \end{cases}$$



XII. Find the inverse Laplace transform of $\ln(s)$.
(3)

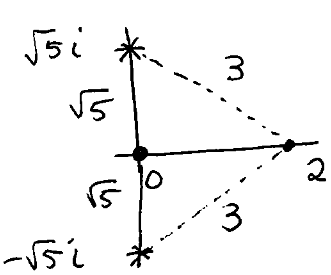
$$\mathcal{L}(tf(t)) = -\frac{d}{ds}\ln(s) = -\frac{1}{s}$$

$$tf(t) = -1$$

$$f(t) = -\frac{1}{t}$$

XIII. For the differential equation $(x^2+5)y'' - 8xy' + 12y = 0$, find the singular points. For solutions of the form $\sum_{n=0}^{\infty} c_n x^n$, how large (at least) is the radius of convergence guaranteed to be? For solutions of the form $\sum_{n=0}^{\infty} c_n (x-2)^n$, how large (at least) is the radius of convergence guaranteed to be?
(4)

$$y'' - \frac{8x}{x^2+5}y' + \frac{12}{x^2+5}y = 0,$$ coefficient functions are nonanalytic only at the singular points $\pm\sqrt{5}\,i$



For $\sum c_n x^n$, $\rho \geq \sqrt{5}$

For $\sum c_n (x-2)^n$, $\rho \geq 3$

XIV. (9) 1. Define what it means to say that the functions $f_1(x), f_2(x), \ldots, f_n(x)$ are *linearly dependent* on the interval I.

It means there exist constants $k_1, k_2, \ldots, k_n$, not all 0, such that $k_1 f_1(x) + k_2 f_2(x) + \ldots + k_n f_n(x) = 0$

2. Write $\dfrac{x^2}{(x+1)^3}$ in terms of partial fractions.

$$\frac{x^2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$x^2 = A(x+1)^2 + B(x+1) + C$, for $x=-1$, $1 = C$

$2x = 2A(x+1) + B$, for $x=-1$, $-2 = B$

$2 = 2A$ so $A = 1$

$$\frac{x^2}{(x+1)^3} = \frac{1}{x+1} + \frac{-2}{(x+1)^2} + \frac{1}{(x+1)^3}$$

3. Using the definition, show that $\dfrac{1}{x+1}, \dfrac{1}{(x+1)^2}, \dfrac{1}{(x+1)^3}$, and $\dfrac{x^2}{(x+1)^3}$ are linearly dependent on the interval $I = (0, \infty)$.

$$1 \cdot \frac{x^2}{(x+1)^3} + (-1)\frac{1}{x+1} + 2 \cdot \frac{1}{(x+1)^2} + (-1)\frac{1}{(x+1)^3} = 0$$

4. Show that if $f(x)$, $g(x)$, and $h(x)$ are three functions that are linearly dependent on an interval I, then $f'(x)$, $g'(x)$, and $h'(x)$ are also linearly dependent on I.

Let $f(x), g(x), h(x)$ be linearly dependent.
There are numbers A, B, C, not all 0, with $Af(x) + Bg(x) + Ch(x) = 0$ on I
Taking derivatives, $Af'(x) + Bg'(x) + Ch'(x) = 0$. on I.
Since not all of A, B, C are 0, $f'(x), g'(x), h'(x)$ are linearly dependent on I.

XV. Consider the boundary-value problem $y'' + \lambda y = 0,\ \ y(a) = a_0,\ y(b) = b_0$.
(2)

1. Define what it means to say that a value of λ *is an eigenvalue* of this problem.

   It means it is a value for which the resulting BVP has a nonzero solution.

2. Define what it means to say that a function y *is an eigenfunction* associated to an eigenvalue λ.

   It means y is a nonzero solution for the BVP obtained for this value of λ.

XVI. Consider the boundary-value problem $y'' + \lambda y = 0,\ \ y'(0) = 0,\ y'(\pi) = 0$.
(6)

1. Show that $\lambda = 0$ is an eigenvalue of this problem.

   $y = 1$ is a nonzero function satisfying

   $y'' = 0$, $y'(0) = 0$, and $y'(\pi) = 0$.

2. By writing $\lambda = \alpha^2$, $\alpha > 0$, find all positive eigenvalues, and an associated eigenfunction for each positive eigenvalue.

$y'' + \alpha^2 y = 0$

$y = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$

$y' = -c_1 \alpha \sin(\alpha x) + c_2 \alpha \cos(x)$

$0 = y'(0) = c_2 \alpha$, so $c_2 = 0$ and $y = c_1 \cos(\alpha x)$

$y' = -c_1 \alpha \sin(\alpha x)$

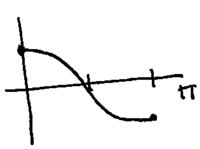
$0 = y'(\pi) = -c_1 \alpha \sin(\alpha \pi)$

$0 = \sin(\alpha \pi)$ so $\alpha \pi = n\pi$, $n = 1, 2, 3, \ldots$

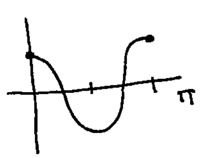
$\alpha = n$, $n = 1, 2, 3$

eigenvalues are n^2 $\quad n = 1, 2, 3, \ldots$

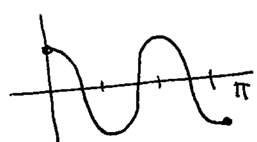
with associated eigenfunctions $\cos(nx)$



$n = 1$



$n = 2$



$n = 3$