Mathematics 3113-001 Name (please print) ____________________

Final Examination Form B

December 15, 2003

I. (4) Let $f(t) = t$ for $0 \le t < \pi^3$, and $f(t) = 0$ for $t \ge \pi^3$. Use the *definition* of the Laplace transform to calculate $\mathcal{L}\{f(t)\}$.

II. (4) Let $f(t) = t$ for $0 \le t < \pi^3$, and $f(t) = 0$ for $t \ge \pi^3$. Use a step function to write an expression for $f(t)$, and use the formulas list for Laplace transforms to calculate $\mathcal{L}\{f(t)\}$.

III. (4) Solve the initial value problem $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = -1$ by using the characteristic equation to find the general solution, then solving equations involving the initial values to find the solution that satisfies the initial values.

IV. (4) Solve the initial value problem $x'' + 4x = 0$, $x(0) = 0$, $x'(0) = -1$ by using the Laplace transform.

V. Use separation of variables to solve $\dfrac{dy}{dx} = 6e^{2x-y}$.
(4)

VI. Solve the initial value problem $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = -1$ as follows.
(8)

1. Put $y = \displaystyle\sum_{n=0}^{\infty} c_n x^n$, and write an expresssion for y'' as a series. Use these expresssions in the differential equation to find a recursive formula for c_n (it should give c_{n+2} in terms of c_n).

2. Use the initial conditions $y(0) = 0$ and $y'(0) = -1$ to find c_0 and c_1, and solve for c_n, treating separately the cases n even and n odd.

VII. Solve the initial value problem $x'' + 4x = 0, \;\; x(0) = 0, \;\; x'(0) = -1$ as follows.
(6)

1. Putting $x_1 = x$ and $x_2 = x'$, rewrite the equation as a system of two first-order equations. Notice that the initial conditions become $x_1(0) = 0$ and $x_2(0) = -1$.

2. Use the Laplace transform and Cramer's rule to find an expression for the Laplace transform $X_1(s)$ of $x_1(t)$, and use the inverse transform to find $x_1(t)$ (of course, there is no need to also find $X_2(s)$ and $x_2(t)$, since $x_1(t)$ is $x(t)$, the solution to the original equation).

VIII. Use an integrating factor to solve the first-order linear equation $y' = (1 - y)\cos(x), \;\; y(\pi) = 2$.
(4)

IX. Write the Maclaurin series for $\sin(x)$, $\dfrac{\sin(x)}{x}$, $\cosh(x)$, and e^{-x^2}.
(4)

X. If D is the differential operator defined by $Df = f'$, calculate $(D^2 + e^x D + e^{2x})(xe^{3x})$.
(5)

XI. Solve $x'' = \delta(t) - \delta(t-1)$, $x(0) = 0$, $x'(0) = 0$, and graph the solution.
(5)

XII. Find the inverse Laplace transform of $\ln(s)$.
(3)

XIII. For the differential equation $(x^2+5)y'' - 8xy' + 12y = 0$, find the singular points. For solutions of the form $\sum_{n=0}^{\infty} c_n x^n$, how large (at least) is the radius of convergence guaranteed to be? For solutions of the form $\sum_{n=0}^{\infty} c_n (x-2)^n$, how large (at least) is the radius of convergence guaranteed to be?
(4)

XIV. 1. Define what it means to say that the functions $f_1(x)$, $f_2(x), \ldots,$ $f_n(x)$ are *linearly dependent* on the
(9) interval I.

2. Write $\dfrac{x^2}{(x+1)^3}$ in terms of partial fractions.

3. Using the definition, show that $\dfrac{1}{x+1}$, $\dfrac{1}{(x+1)^2}$, $\dfrac{1}{(x+1)^3}$, and $\dfrac{x^2}{(x+1)^3}$ are linearly dependent on the interval $I = (0, \infty)$.

4. Show that if $f(x)$, $g(x)$, and $h(x)$ are three functions that are linearly dependent on an interval I, then $f'(x)$, $g'(x)$, and $h'(x)$ are also linearly dependent on I.

XV. Consider the boundary-value problem $y'' + \lambda\, y = 0,\;\; y(a) = a_0,\; y(b) = b_0.$
(2)

1. Define what it means to say that a value of λ *is an eigenvalue* of this problem.

2. Define what it means to say that a function y *is an eigenfunction* associated to an eigenvalue λ.

XVI. Consider the boundary-value problem $y'' + \lambda y = 0,\;\; y'(0) = 0,\; y'(\pi) = 0.$
(6)

1. Show that $\lambda = 0$ is an eigenvalue of this problem.

2. By writing $\lambda = \alpha^2$, $\alpha > 0$, find all positive eigenvalues, and an associated eigenfunction for each positive eigenvalue.