Theory of Second-Order Linear Ordinary Differential Equations

1. A **second-order linear equation** is a differential equation of the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$$

(where $P_0(X)$, $P_1(x)$, $P_2(x)$, and $F(x)$ are continuous). At x-values where $P_0(x) = 0$, the behavior is complicated. On any open interval I where $P_0(x)$ is never 0, we can divide by $P_0(x)$ to obtain the general equation

$$y'' + p_1(x)y' + p_2(x)y = f(x) \ .$$

This equation is called **homogeneous** if $f(x) = 0$, otherwise it is called **nonhomogeneous.** From now on, we will *assume* that these functions $p_1(x)$, $p_2(x)$ and $f(x)$ are continuous on some open interval I.

2. For the *homogeneous* equation $y'' + p_1(x)y' + p_2(x)y = 0$, we have the **Principle of Superposition**: if $y_1, \ldots, y_r$ are solutions, then so is any linear combination $k_1y_1 + \cdots + k_ry_r$.

3. **Existence and Uniqueness**: For any number a in the interval I, if b_0 and b_1 are any real numbers then the *initial value problem*

$$y'' + p_1(x)y' + p_2(x)y = f(x); \quad y(a) = b_0, y'(a) = b_1$$

has a *unique* solution which is *defined on all of I.*

4. A pair of functions f_1 and f_2 on the interval I is called **linearly dependent** if there are constants k_1 and k_2, at least one of which is not 0, so that $k_1f_1 + k_2f_2 = 0$ (for all x in I). This happens exactly when you can express f_1 or f_2 as a constant multiple of the other one. For example, if $k_1 \neq 0$, then you can solve for f_1 to obtain $f_1 = -\frac{k_2}{k_1}f_2$. If the pair of functions is not linearly dependent, it is called **linearly independent.**

5. The **Wronskian** of the pair f_1, f_2 is the *function* which is the determinant

$$W(f_1, f_2) = \det \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} \ .$$

If f_1, f_2 are linearly dependent on I then $W(f_1, f_2)$ is the zero function.

If f_1, f_2 are linearly independent *solutions* of the homogeneous linear equation

$$y'' + p_1(x)y' + p_2(x)y = 0$$

on I, then $W(f_1, f_2)(x)$ is not zero *for any x in I.*

6. **General Solution for a Second-Order Homogeneous Linear Equation**: If y_1 and y_2 are linearly independent solutions of the *homogeneous* equation $y'' + p_1(x)y' + p_2(x)y = 0$, then *every* solution is a linear combination $y_c = c_1y_1 + c_2y_2$.

7. **General Solution for a Second-Order Nonhomogeneous Linear Equation**: If y_p is a particular solution of the *nonhomogeneous* equation $y'' + p_1(x)y' + p_2(x)y = f(x)$, then *every* solution is a linear combination $y_p + y_c$ where y_c is some solution of the associated homogeneous equation $y'' + p_1(x)y' + p_2(x)y = 0$.