*Theory of Linear Ordinary Differential Equations*

1. The **general linear equation** is

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots P_{n-1}(x)y' + P_n(x)y = F(x)\ .$$

The number $n$ is called the *order* of the equation. At $x$-values where $P_0(x) = 0$, the behavior is complicated. On any open interval $I$ where $P_0(x)$ is never 0, we can divide by $P_0(x)$ to obtain the general equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)\ .$$

This equation is called **homogeneous** if $f(x) = 0$, otherwise it is called **nonhomogeneous.** From now on, we will *assume* that these functions $p_1(x), \ldots, p_n(x)$ and $f(x)$ are continuous on some open interval $I$.

2. For the *homogeneous* equation $y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$, we have the **Principle of Superposition**: if $y_1, \ldots, y_r$ are solutions, then so is any linear combination $k_1y_1 + \cdots + k_ry_r$.

3. **Existence and Uniqueness**: For any number $a$ in the interval $I$, if $b_0, b_1, \ldots, b_{n-1}$ are any real numbers then the *initial value problem*

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x);\ \ y(a) = b_0, y'(a) = b_1, \ldots, y^{(n-1)}(a) = b_{n-1}$$

has a *unique* solution which is *defined on all of $I$.*

4. A collection of functions $f_1, \ldots, f_r$ on the interval $I$ is called **linearly dependent** if there are constants $k_1, \ldots, k_r$, at least one of which is not 0, so that $k_1f_1 + \cdots + k_rf_r = 0$ (for all $x$ in $I$). This happens exactly when you can express one of the $f_i$ as a linear combination of the others. For example, if $k_1 \neq 0$, then you can solve for $f_1$ to obtain $f_1 = -\frac{k_2}{k_1}f_2 - \frac{k_3}{k_1}f_3 - \cdots - \frac{k_r}{k_1}f_r$. If the set of functions is not linearly dependent, it is called **linearly independent.**

5. The **Wronskian** of the collection $f_1, \ldots, f_n$ is the *function* which is the determinant

$$W(f_1, \ldots, f_n) = \det \begin{pmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ f_1' & f_2' & f_3' & \cdots & f_n' \\ f_1'' & f_2'' & f_3'' & \cdots & f_n'' \\ \vdots & \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix}\ .$$

If $f_1, \ldots\ f_n$ are linearly dependent on $I$ then $W(f_1, \ldots, f_n)$ is the zero function.

If $f_1, \ldots\ f_n$ are linearly independent *solutions* of the homogeneous linear equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$$

on $I$, then $W(f_1, \ldots, f_n)(x)$ is not zero *for any $x$ in $I$.*

6. **General Solution for a Homogeneous Linear Equation**: If $y_1, \ldots, y_n$ are linearly independent solutions of the *homogeneous* equation $y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$, then *every* solution is a linear combination $y_c = c_1y_1 + \cdots + c_ny_n$.

7. **General Solution for a Nonhomogeneous Linear Equation**: If $y_p$ is a particular solution of the *nonhomogeneous* equation $y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)$, then *every* solution is a linear combination $y_p + y_c$ where $y_c$ is some solution of the associated homogeneous equation $y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$.