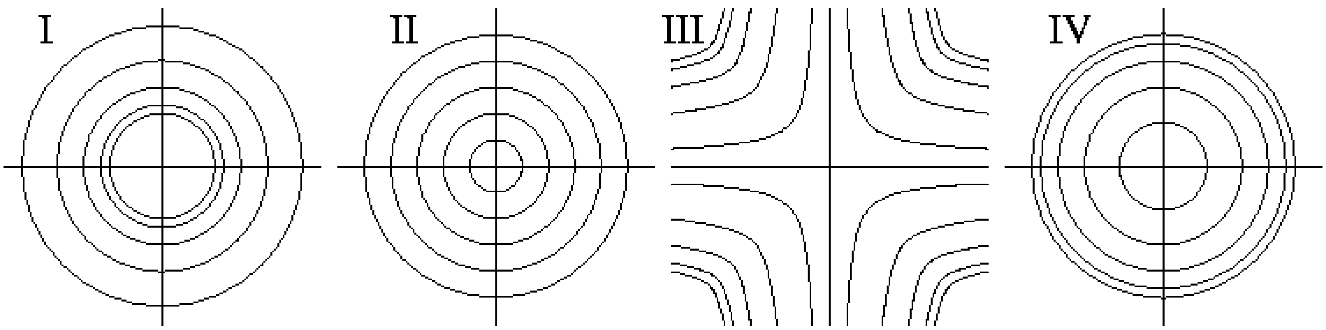
Mathematics 2443-006 Name (please print) ______________________

Final Examination

December 13, 2001

I. (4) The coordinate systems below show the level curves for *equally spaced values* of four functions. Write the Roman numeral of the coordinate system front of its function.

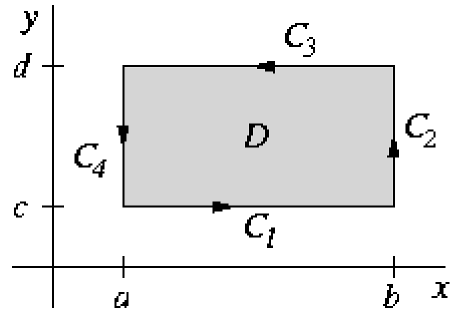
1. _____ $\sqrt{x^2+y^2}$
2. _____ $\dfrac{1}{x^2+y^2}$
3. _____ x^2+y^2
4. _____ xy



II. (3) State the Fundamental Theorem for Line Integrals.

III. (4) Verify that $\ln(x+at)$ is a solution to the wave equation, $u_{tt}=a^2u_{xx}$.

IV. (4) Let $P(x, y)$ and $Q(x, y)$ be functions with continuous partial derivatives, defined on the rectangle D shown to the right. The four sides of D are C_1, C_2, C_3, and C_4. Verify the following facts. (Do not try to use Green's Theorem; it is not the best way to verify these facts, and that approach would not be appropriate anyway, since these facts are some of the key steps in the proof of Green's Theorem).



1. $\displaystyle\iint_D \frac{\partial Q}{\partial x}\,dA = \int_c^d Q(b, y)\,dy - \int_c^d Q(a, y)\,dy$

2. $\displaystyle\int_{C_2} P\,dx + Q\,dy = \int_c^d Q(b, y)\,dy$

V. (8) Supply limits of integration for the following triple integrals.

1. $\iiint_E f(x, y, z)\,dV$, where E is the solid tetrahedron with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(0, 3, 0)$, $(0, 0, 4)$.

2. $\iiint_E f(\rho, \theta, \phi)\,dV$, where E is the region inside the sphere of radius 3 and in the octant where $x \leq 0$, $y \geq 0$, and $z \leq 0$.

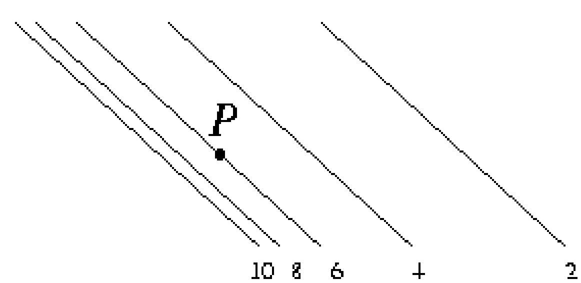
VI. (9) A certain curve C is parameterized by $x = 2\sqrt{t}$, $y = t$, and $z = t^2$ for $1 \leq t \leq 2$. For this curve, calculate the following line integrals.

1. $\int_C \left(8y - \dfrac{1}{z}\right) ds$

2. $\int_C (z\vec{\jmath} + y\vec{k}) \cdot d\vec{r}$, by direct calculation

3. $\int_C (z\vec{\jmath} + y\vec{k}) \cdot d\vec{r}$, using the Fundamental Theorem for Line Integrals

VII. (4) Some level lines of a certain function $g(x, y)$ near a point P are shown to the right. Answer the following, assuming the most likely behavior of g indicated by the values of g on these level lines.



1. Is $\dfrac{\partial g}{\partial y}$ positive, negative, or zero?

2. Is $\dfrac{\partial^2 g}{\partial x^2}$ positive, negative, or zero?

3. Draw and label ∇g at P.

4. Draw and label a direction at P in which the directional derivative of P is very slightly less than zero.

VIII. (6) Use Stokes' Theorem to calculate $\int_C ((x+y^2)\vec{\imath} + (y+z^2)\vec{\jmath} + (z+x^2)\vec{k}) \cdot d\vec{r}$, where C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ oriented counterclockwise as viewed from above (take S to be the portion of the plane $x+y+z=1$ that lies in the first octant).

IX. (6) Calculate that $\mathrm{curl}(xy\vec{\imath}+yz\vec{\jmath}+xz\vec{k}) = -y\vec{\imath}-z\vec{\jmath}-x\vec{k}$. Use Stokes' Theorem to calculate $\iint_S (-y\vec{\imath}-z\vec{\jmath}-x\vec{k})\cdot d\vec{S}$, where S is the hemisphere $z=\sqrt{a^2-x^2-y^2}$, $z \geq 0$ oriented upward.

X. Verify the Divergence Theorem for the vector field $\vec{F}(x, y, z) = x\vec{\imath} + y\vec{\jmath}$ and the surface S which is the sphere $x^2 + y^2 + z^2 = 1$.
(7)

XI. Let $g(x, y) = x^3 + xy^2$.
(4)

1. Calculate the gradient ∇g.

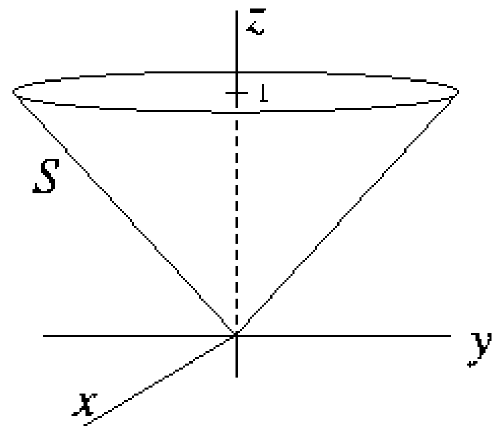
2. Calculate the directional derivative of g at the point $(2, 3)$, in the direction from $(2, 3)$ to $(-1, 6)$.

XII. The length and width of a box are each increasing at 2 cm/sec, and its height is decreasing at 1 cm/sec. How fast is the length of its diagonal changing when the length is 10 and the width and height are 20?
(5)

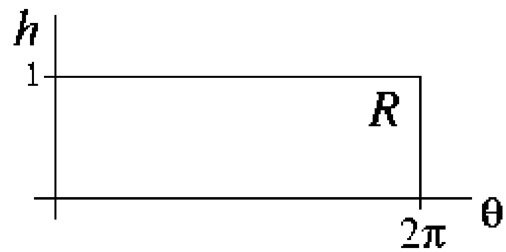
XIII. The picture at the right shows a parameterization of the cone $z = \sqrt{x^2 + y^2}$. It is parameterized by letting θ be the polar angle in the xy-plane, and h be the z-coordinate. The parameterization is
(6)

$$\begin{aligned} x &= h\cos(\theta) \\ y &= h\sin(\theta) \\ z &= h, \end{aligned}$$

where the parameter domain R in the θh-plane consists of $0 \le \theta \le 2\pi$, $0 \le h \le 1$.



1. Calculate $\vec{r}_h$ and $\vec{r}_\theta$.



2. For the line $0 \le h \le 1$, $\theta = \dfrac{\pi}{4}$ in R, draw the corresponding points on S. Do the same for the line $h = \dfrac{1}{2}$, $0 \le \theta \le \dfrac{\pi}{2}$. At the intersection point of these two curves on S, draw the vectors $\vec{r}_\theta$ and $\vec{r}_h$.
3. Is the upward normal equal to $\vec{r}_\theta \times \vec{r}_h$ or to $\vec{r}_h \times \vec{r}_\theta$?

XIV. Find the maximum value of $x^2 - y^2$ on the disc $D = \{(x, y) \mid x^2 + y^2 \le 1\}$ as follows.
(5)

1. Find the critical point or points of $f(x, y)$ on D, and compute the value of $f(x, y)$ at each critical point.
2. Parameterize the boundary of D, and express $f(x, y)$ on the boundary of D in terms of this parameter. Find the maximum value of $f(x, y)$ on the boundary, and where it occurs.
3. Compare the values found in the first two steps to determine the actual maximum value.