Mathematics 2443-002 Name (please print) ______________________

Final Exam Form B

December 15, 2000

I. Rewrite the following integrals using polar or spherical coordinates. Be sure to supply the correct limits
(8) of integration for the new variables. *Do not* carry out the calculation of the integrals.

1. $\int_0^1 \int_{-\sqrt{1-x^2}}^0 e^{x^2+y^2}\, dy\, dx$

2. $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$

II. A curve C is parameterized as $\vec{r}(t) = e^t\,\vec{\imath} + \ln(\cos(t))\,\vec{\jmath}$ for $0 \le t \le 1$. Using this parameterization,
(6) express each of the following as an ordinary definite integral of a function of t. Make reasonable algebraic simplifications, but *do not* try to calculate the actual values of the integrals.

1. $\int_C \ln(x)e^y\, ds$

2. $\int_C e^x\, dy$

3. $\int_C (x^2\vec{\imath} + e^y\vec{\jmath}) \cdot d\vec{r}$

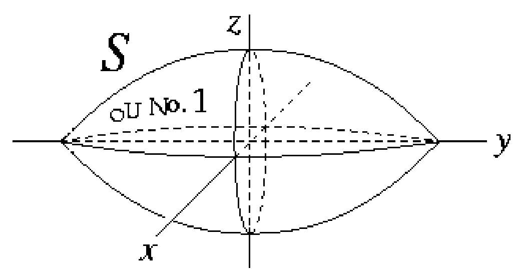
III. (4) Let $\vec{F}(x,y,z) = -xz\vec{\imath} - yz\vec{\jmath} + z^2\vec{k}$, and let the surface S be the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Calculate $\iint_S \vec{F} \cdot d\vec{S}$, where S has the outward normal.

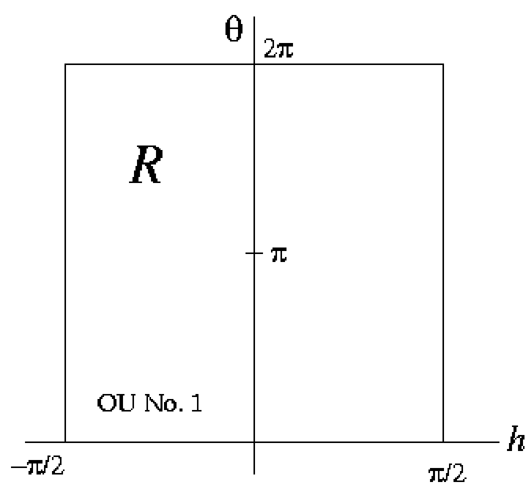
IV. (5) Let r and θ be the polar coordinates in the xy-plane. For the function $f(x,y) = x^3y$, calculate $\frac{\partial f}{\partial \theta}$ in terms of x and y.

V. (15) The picture at the right shows a parameterization of the outer surface S of a football. It is parameterized by letting θ be the polar angle in the xz-plane, and h be the y-coordinate. The parameterization is

$$\begin{aligned} x &= \cos(h)\cos(\theta) \\ y &= h \\ z &= \cos(h)\sin(\theta), \end{aligned}$$

where the parameter domain R in the $h\theta$-plane consists of $\frac{\pi}{2} \le h \le \frac{\pi}{2}$, $0 \le \theta \le 2\pi$.





1. Calculate $\vec{r}_h$ and $\vec{r}_\theta$.

2. For the line $0 \le h \le \frac{\pi}{2}$, $\theta = \frac{\pi}{4}$ in R, draw the corresponding points on S. Do the same for the line $h = \frac{\pi}{4}$, $0 \le \theta \le \frac{\pi}{2}$. At the intersection point of these two curves on S, draw the vectors $\vec{r}_\theta$ and $\vec{r}_h$.

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(This problem is continued from the previous page.)

3. Given that $\vec{r}_h \times \vec{r}_\theta = \cos(h)\cos(\theta)\vec{\imath} + \sin(h)\cos(h)\vec{\jmath} + \cos(h)\sin(\theta)\vec{k}$, carry out the following:

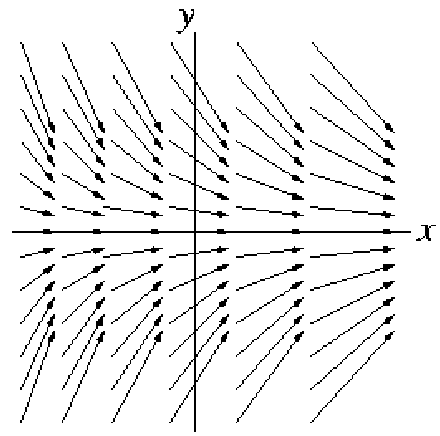
   (a) Check that $\| \vec{r}_h \times \vec{r}_\theta \| = \cos(h)\sqrt{1+\sin^2(h)}$.

   (b) Calculate $\iint_S \sqrt{1+\sin^2(y)}\, dS$.

   (c) Calculate $\iint_S \sin(y)\cos(y)\vec{\jmath} \cdot d\vec{S}$, where the outward normal is used on S.

   (d) Given that the volume of the football is $\frac{\pi^2}{2}$, calculate the flux of the vector field $(x+y)\vec{\imath}+(y+z)\vec{\jmath}+(z+x)\vec{k}$ across S, where the outward normal is used on S.

VI. The picture at the right shows a certain vector field $P\vec{\imath}+Q\vec{\jmath}$
(6) in the xy-plane.



1. Is $\dfrac{\partial P}{\partial x}$ positive, negative, or zero?

2. Is $\dfrac{\partial P}{\partial y}$ positive, negative, or zero?

3. Is $\dfrac{\partial Q}{\partial x}$ positive, negative, or zero?

4. Is $\dfrac{\partial Q}{\partial y}$ positive, negative, or zero?

5. If C is the line segment from $(1,0)$ to $(1,1)$, is $\int_C (P\vec{\imath}+Q\vec{\jmath})\cdot d\vec{r}$ positive, negative, or zero?

6. If C is the line segment from $(1,-1)$ to $(1,0)$, is $\int_C (P\vec{\imath}+Q\vec{\jmath})\cdot d\vec{r}$ positive, negative, or zero?

VII. Let $f(x,y,z)$ be the function ze^{xy^2}.
(8)

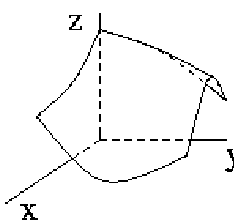
1. Calculate the gradient $\nabla f(x,y,z)$.

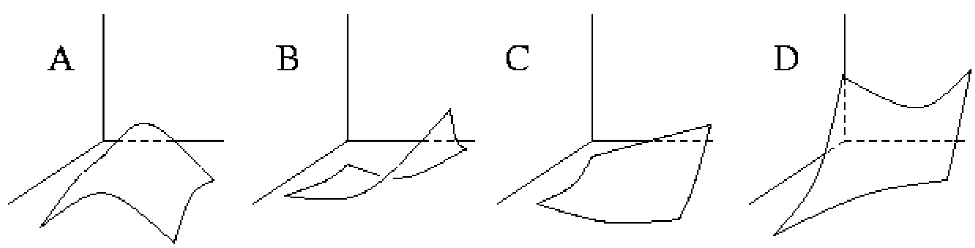
2. Calculate the rate of change of f at the point $(3,1,1)$, in the direction toward the point $(0,0,2)$.

3. Let S be the level surface of f through the point $(3,1,1)$. Find an equation for the tangent plane to S at the point $(3,1,1)$.

VIII. Let C consist of the clockwise arc of the circle $x^2 + y^2 = 4$ from $(2, 0)$ to $(0, 2)$, followed by the line segment
(7) from $(0, 2)$ to $(2, 4)$. Calculate $\int_C (2xy\cos(x^2)\,\vec{\imath} + (\sin(x^2) + \cos(y)\,\vec{\jmath}) \cdot d\vec{r}$.

IX. The following two questions refer to the function $f(x, y)$ whose graph is shown here, and the four functions
(4) indicated in the graphs A, B, C, and D. All of the functions have domain the points (x, y) with $0 \leq x \leq 1$ and $0 \leq y \leq 1$.





1. The graph of the function $\dfrac{\partial f}{\partial y}$ looks most like the function in

   (a) A (b) B (c) C (d) D

2. The graph of the function $\dfrac{\partial f}{\partial x}$ looks most like the function in

   (a) A (b) B (c) C (d) D

X. (10) Let T be the part of the plane $x+y+z=1$ that lies in the first octant, with the upward normal, and let C be its boundary, with the positive orientation.

1. Use Stokes' Theorem to write $\int_C (xz\,\vec{\imath}+3xy\,\vec{\jmath}+3xy\,\vec{k})\cdot d\vec{r}$ as a surface integral on T.

2. Regarding T as the graph of the function $z=1-x-y$ over a domain D in the xy-plane, use the formula $\iint_S (P\,\vec{\imath}+Q\,\vec{\jmath}+R\,\vec{k})\cdot d\vec{S}=\iint_D -P\,g_x-Q\,g_y+R\,dD$ to calculate the surface integral.

XI. (3) Let D be a domain in the plane. Define what it means to say that D is *simply-connected.*