I. Rewrite the following integrals using polar or spherical coordinates. Be sure to supply the correct limits of integration for the new variables. Do not carry out the calculation of the integrals.

1. $\int_{0}^{1} \int_{\sqrt{1-x^2}}^{0} e^{x^2+y^2} \, dy \, dx$

2. $\int_{0}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{\sqrt{9-x^2-y^2}} z \sqrt{x^2+y^2+z^2} \, dz \, dy \, dx$

II. A curve $C$ is parameterized as $\vec{r}(t) = e^t \vec{i} + \ln(\cos(t)) \vec{j}$ for $0 \leq t \leq 1$. Using this parameterization, express each of the following as an ordinary definite integral of a function of $t$. Make reasonable algebraic simplifications, but do not try to calculate the actual values of the integrals.

1. $\int_{C} \ln(x) e^y \, ds$

2. $\int_{C} e^x \, dy$

3. $\int_{C} (x^2 \vec{i} + e^y \vec{j}) \cdot d\vec{r}$
III. Let \( \vec{F}(x, y, z) = -xz \hat{i} - yz \hat{j} + z^2 \hat{k} \), and let the surface \( S \) be the ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \). Calculate \( \iint_S \vec{F} \cdot d\vec{S} \), where \( S \) has the outward normal.

IV. Let \( r \) and \( \theta \) be the polar coordinates in the \( xy \)-plane. For the function \( f(x, y) = x^3y \), calculate \( \frac{\partial f}{\partial \theta} \) in terms of \( x \) and \( y \).

V. The picture at the right shows a parameterization of the outer surface \( S \) of a football. It is parameterized by letting \( \theta \) be the polar angle in the \( xz \)-plane, and \( h \) be the \( y \)-coordinate. The parameterization is

\[
\begin{align*}
x &= \cos(h) \cos(\theta) \\
y &= h \\
z &= \cos(h) \sin(\theta),
\end{align*}
\]

where the parameter domain \( R \) in the \( h\theta \)-plane consists of \( \frac{\pi}{2} \leq h \leq \frac{\pi}{2}, \ 0 \leq \theta \leq 2\pi \).

1. Calculate \( \vec{r}_h \) and \( \vec{r}_\theta \).

2. For the line \( 0 \leq h \leq \frac{\pi}{2}, \ \theta = \frac{\pi}{4} \) in \( R \), draw the corresponding points on \( S \). Do the same for the line \( h = \frac{\pi}{4}, \ 0 \leq \theta \leq \frac{\pi}{2} \). At the intersection point of these two curves on \( S \), draw the vectors \( \vec{r}_\theta \) and \( \vec{r}_h \).

(This problem is continued on the next page)
3. Given that \( \vec{r}_h \times \vec{r}_\theta = \cos(h) \cos(\theta) \vec{i} + \sin(h) \cos(\theta) \vec{j} + \cos(h) \sin(\theta) \vec{k} \), carry out the following:

(a) Check that \( \| \vec{r}_h \times \vec{r}_\theta \| = \cos(h) \sqrt{1 + \sin^2(h)} \).

(b) Calculate \( \iint_S \sqrt{1 + \sin^2(y)} \, dS \).

(c) Calculate \( \iint_S \sin(y) \cos(y) \, dS \), where the outward normal is used on \( S \).

(d) Given that the volume of the football is \( \frac{\pi^2}{2} \), calculate the flux of the vector field \( (x+y) \vec{i} + (y+z) \vec{j} + (z+x) \vec{k} \) across \( S \), where the outward normal is used on \( S \).
VI. The picture at the right shows a certain vector field $P\mathbf{i} + Q\mathbf{j}$ in the $xy$-plane.

1. Is $\frac{\partial P}{\partial x}$ positive, negative, or zero?
2. Is $\frac{\partial P}{\partial y}$ positive, negative, or zero?
3. Is $\frac{\partial Q}{\partial x}$ positive, negative, or zero?
4. Is $\frac{\partial Q}{\partial y}$ positive, negative, or zero?
5. If $C$ is the line segment from $(1, 0)$ to $(1, 1)$, is $\int_C (P\mathbf{i} + Q\mathbf{j}) \cdot d\mathbf{r}$ positive, negative, or zero?
6. If $C$ is the line segment from $(1, -1)$ to $(1, 0)$, is $\int_C (P\mathbf{i} + Q\mathbf{j}) \cdot d\mathbf{r}$ positive, negative, or zero?

VII. Let $f(x, y, z)$ be the function $ze^{xy^2}$.

1. Calculate the gradient $\nabla f(x, y, z)$.
2. Calculate the rate of change of $f$ at the point $(3, 1, 1)$, in the direction toward the point $(0, 0, 2)$.
3. Let $S$ be the level surface of $f$ through the point $(3, 1, 1)$. Find an equation for the tangent plane to $S$ at the point $(3, 1, 1)$.
VIII. Let $C$ consist of the clockwise arc of the circle $x^2 + y^2 = 4$ from $(2, 0)$ to $(0, 2)$, followed by the line segment from $(0, 2)$ to $(2, 4)$. Calculate $\int_C (2xy \cos(x^2) \mathbf{i} + (\sin(x^2) + \cos(y)) \mathbf{j}) \cdot d\mathbf{r}$.

IX. The following two questions refer to the function $f(x, y)$ whose graph is shown here, and the four functions indicated in the graphs A, B, C, and D. All of the functions have domain the points $(x, y)$ with $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

1. The graph of the function $\frac{\partial f}{\partial y}$ looks most like the function in
   (a) A   (b) B   (c) C   (d) D

2. The graph of the function $\frac{\partial f}{\partial x}$ looks most like the function in
   (a) A   (b) B   (c) C   (d) D
X. Let $T$ be the part of the plane $x + y + z = 1$ that lies in the first octant, with the upward normal, and let $C$ be its boundary, with the positive orientation.

1. Use Stokes’ Theorem to write $\int_C (xz \mathbf{i} + 3xy \mathbf{j} + 3xy \mathbf{k}) \cdot d\mathbf{r}$ as a surface integral on $T$.

2. Regarding $T$ as the graph of the function $z = 1 - x - y$ over a domain $D$ in the $xy$-plane, use the formula
   \[ \iint_S (P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}) \cdot d\mathbf{S} = \iint_D -P g_x - Q g_y + R d\mathbf{D} \]
   to calculate the surface integral.

XI. Let $D$ be a domain in the plane. Define what it means to say that $D$ is *simply-connected*.

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