Constructing knot tunnels using giant steps

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A *tunnel number 1 knot* $K \subset S^3$ is a knot for which you can take a regular neighborhood of the knot and add a 1-handle in some way to get an unknotted handlebody (i.e. a handlebody which can be moved by isotopy to the standard handlebody $H$ in $S^3$).

The added 1-handle is called a *tunnel* of $K$. 
An isotopy taking the knot and tunnel to $H$ carries the cocore 2-disk to some *nonseparating* disk $\tau$ in $H$.

And each nonseparating disk $\tau$ in $H$ is the cocore disk of a tunnel of the knot $K_\tau$ which is the core circle of the solid torus obtained by cutting $H$ along $\tau$.

The nonseparating disks in $H$ are the vertices of the *disk complex* $D(H)$. Vertices span a simplex exactly when the corresponding disks are isotopic to a disjoint collection.
\( \mathcal{D}(H) \) looks like this, with countably many 2-simplices meeting at each edge:

![Diagram](image)

and it deformation retracts to the tree \( T \) shown in this figure.

Any two disks in \( H \) coming from equivalent tunnels must differ by an isotopy that moves \( H \) around in \( S^3 \), back to where it started. That is, they differ by the action of an element of the Goeritz group, denoted by \( \mathcal{G} \).

So the collection of all tunnels of all tunnel number 1 knots corresponds to the set of vertices of the quotient complex \( \mathcal{D}(H)/\mathcal{G} \).
Using recent work of M. Scharlemann, E. Akbas, and S. Cho on the genus-2 Goeritz group, it is not hard to work out exactly what $D(H)/G$ looks like:

\[
\pi_0
\]

$\pi_0$ is the orbit of “primitive” disks, which represents the tunnel of the trivial knot.
Moving through $\mathcal{D}(H)/\mathcal{G}$ in different ways corresponds to geometric constructions of new tunnels from old ones. Here is the first way, the “cabling construction.”

Fix a tunnel $\tau$.

$T/\mathcal{G}$ is a tree. The unique path in $T/\mathcal{G}$ from the “root” of $T/\mathcal{G}$ to the nearest barycenter of a simplex that contains $\tau$ is called the \textit{principal path} of $\tau$:
Traveling along the principal path of $\tau$ encodes a sequence of simple cabling constructions, starting with the tunnel of the trivial knot and ending with $\tau$.

The following picture indicates how this works:

Since $T/G$ is a tree, every tunnel can be obtained by starting from $\pi_0$ and performing a unique sequence of cabling constructions.
Moving through the 1-skeleton of $\mathcal{D}(H)/\mathcal{G}$ corresponds to a geometric construction of tunnels that first appeared in a paper of H. Goda, M. Scharlemann, and A. Thompson in 2000. We call it a giant step (Giant Step).

Start with a knot and a tunnel $\tau$.

(This is a picture up to abstract homeomorphism of $H$. In $S^3$, the picture usually looks much more complicated.)
Choose any loop $K$ in $\partial H$ that crosses $\tau$ in exactly one point. It turns out that this must be a tunnel number 1 knot with a tunnel disk $\sigma$ disjoint from $\tau$.

In $\mathcal{D}(H)/\mathcal{G}$, this giant step corresponds to moving along the 1-simplex from $\tau$ to $\sigma$.
This example $\tau$ can be obtained from the trivial tunnel by 5 giant steps.

Giant steps can have a much more drastic effect than cabling constructions—this example requires 15 cabling constructions. Also, any $(1,1)$-tunnel is produced from the trivial tunnel by a single giant step.
Unlike the cabling sequence, a minimal giant step sequence producing a given tunnel is usually not unique. In this example, there are two places where another route is possible, leading to four possible minimal giant step sequences producing $\tau$.

We will now describe a general algorithm to compute the number of minimal paths from $\pi_0$ to $\tau$ in the 1-skeleton of $\mathcal{D}(H)/\mathcal{G}$, and hence the number of minimal giant step constructions of a tunnel.
The simplices that meet the principal path of $\tau$ form the corridor of $\tau$:

The “distance-from-$\pi_0$” (or “depth”) function breaks the corridor into blocks, each having one of four types:
The type of the \(i^{th}\) block determines a matrix \(M_i\) given in this table:

<table>
<thead>
<tr>
<th></th>
<th>(L_1)</th>
<th>(R_1)</th>
<th>(L_2)</th>
<th>(R_2)</th>
</tr>
</thead>
</table>
| \(M_i\) | \[
\begin{pmatrix}
1 & 0 \\
1 & 1 \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & 0 \\
1 & 1 \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & 1 \\
0 & 0 \\
\end{pmatrix}
\] |

The number of distinct minimal giant step sequences can be worked out easily from the entries of the product

\[M_2M_3\cdots M_n\,.
\]
The algorithm is easy to implement computationally.

The input is a binary string $s_2 s_3 \cdots s_n$ which describes the structure of the corridor (roughly speaking, $s_i = 0$ means "go horizontally", $s_i = 1$ means "go down to the next larger depth").

For our previous example, the input string is 0011100011100.

```
Depth> gst('0011100011100', verbose=True)
```

The intermediate configurations are L1, R2, R1.

The transformation matrices are:

\[
\begin{bmatrix}
1 & 0 \\
1 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 \\
0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 \\
0 & 1 \\
\end{bmatrix}
\]

and their product is \[
\begin{bmatrix}
1 & 2 \\
1 & 2 \\
\end{bmatrix}
\].

The final block has configuration L2.

This tunnel has 4 minimal giant step constructions.
Examples:

This corridor corresponds to the parameter sequence 1010101, and there are 8 minimal giant step constructions. An example of a tunnel with this corridor is the “middle” tunnel of the (99, 70) torus knot.

In general, for the sequence $s_2s_3\cdots s_{2n} = 1010\cdots101$, the number of minimal giant step sequences is the term $F_{n+2}$ in the Fibonacci sequence $(F_1, F_2, F_3, \ldots) = (1, 1, 2, 3, 5, \ldots)$. 
1. \( s_2 s_3 \cdots s_{2n+1} = 111 \cdots 1 \), an even number of 1’s. There is a unique minimal giant step sequence.

2. \( s_2 s_3 \cdots s_{2n} = 111 \cdots 1 \), an odd number of 1’s. There are \( n + 1 \) minimal giant step sequences.

Examples of these two types differ by a single additional cabling construction.

For a sparse infinite set of tunnels, there is a unique minimal giant step sequence.

A randomly chosen tunnel will have many minimal giant step sequences.