Disk complexes, arc complexes, and knots

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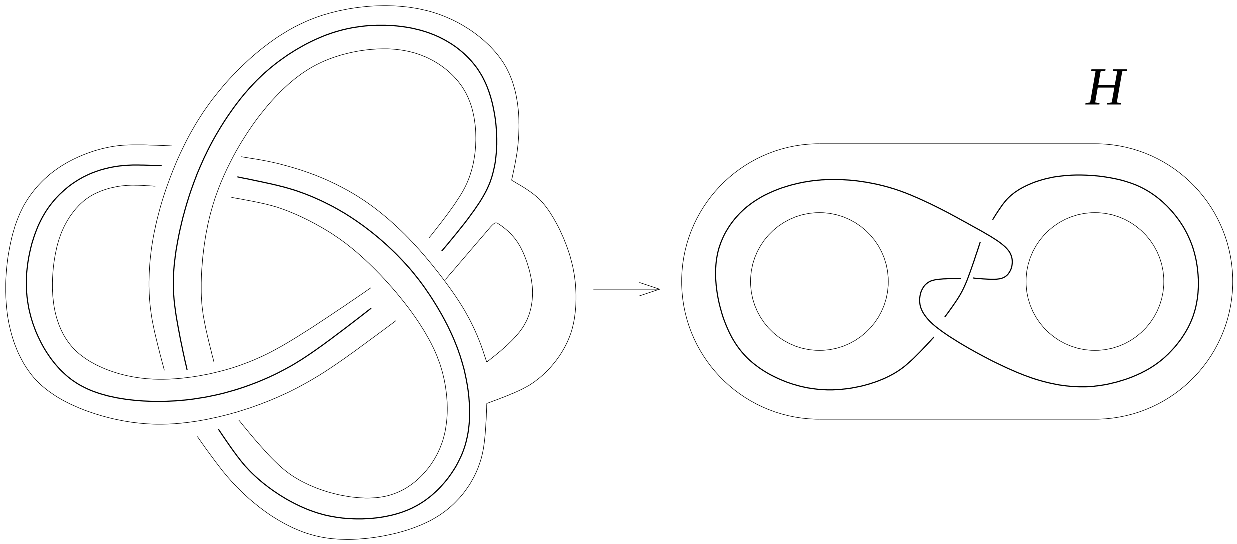
Topics:

I. *The tree of knot tunnels:* a classification of all tunnels of all tunnel number 1 knots (or equivalently of all genus-2 Heegaard splittings of exteriors of knots in S^3), using the disk complex of the genus-2 handlebody (joint with Sangbum Cho).

II. *Depth and bridge numbers:* the "depth" invariant obtained from the classification, and its application to bridge numbers of tunnel number 1 knots (joint with Sangbum Cho).

III. *Level position of knots:* a new application of arc complexes to knot theory (joint with Sangbum Cho and Arim Seo).

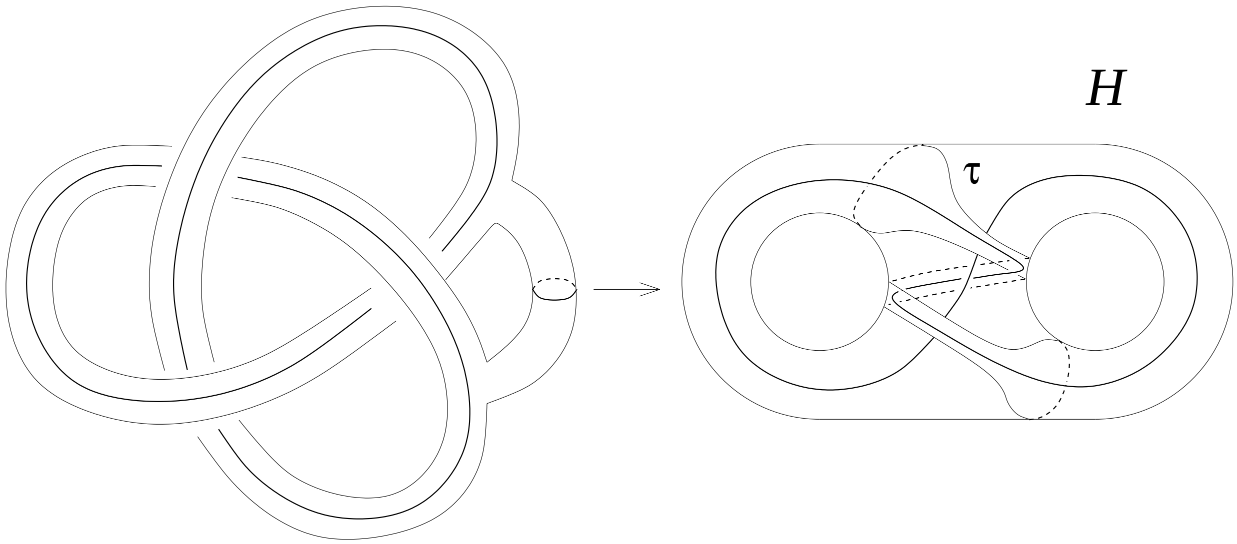
The classic picture:



H = standard genus-2 handlebody in S^3

a tunnel of a tunnel number 1 knot
(up to o. p. homeomorphism)

= a genus-2 Heegaard splitting
of a knot exterior
(up to o. p. homeomorphism)



Under an isotopy moving the neighborhood of the knot and the tunnel to the standard handlebody H, the cocore disk of the tunnel moves to a disk τ in H.

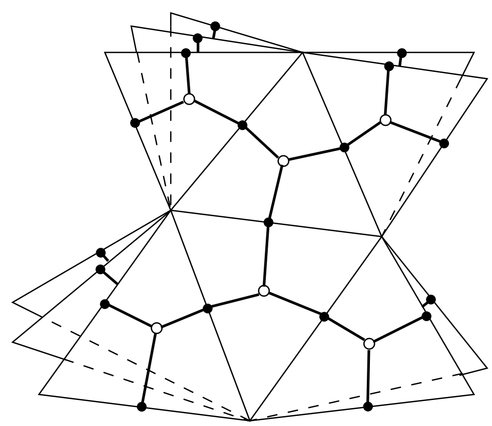
τ is well-defined up to a homeomorphism of H that results from moving H by isotopy through S^3 and back to its standard position.

The group of such homeomorphisms of H is called the (genus-2) *Goeritz group* $\mathcal{G}$.

($\mathcal{G}$ equals the group of isotopy classes of orientation-preserving homeomorphisms of H that extend to S^3.)

We can use this viewpoint to describe all the tunnels of tunnel number 1 knots, using the complex $\mathcal{D}(H)$ of nonseparating disks in H.

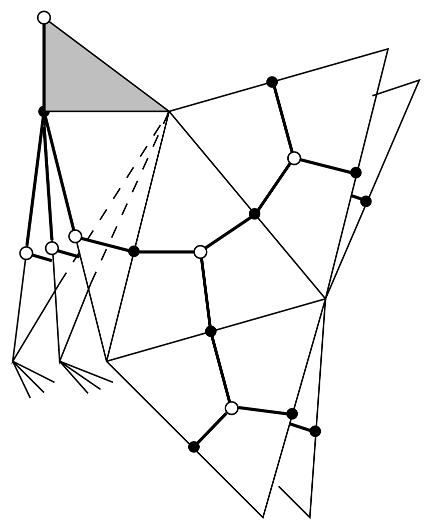
$\mathcal{D}(H)$ looks like this, with countably many 2-simplices meeting at each edge:



and it deformation retracts to the tree T shown in this figure.

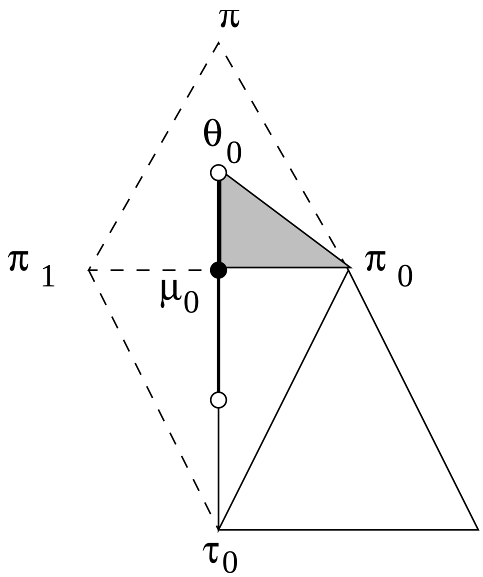
Each white vertex of T is a triple of nonseparating disks, and each black vertex is a pair.

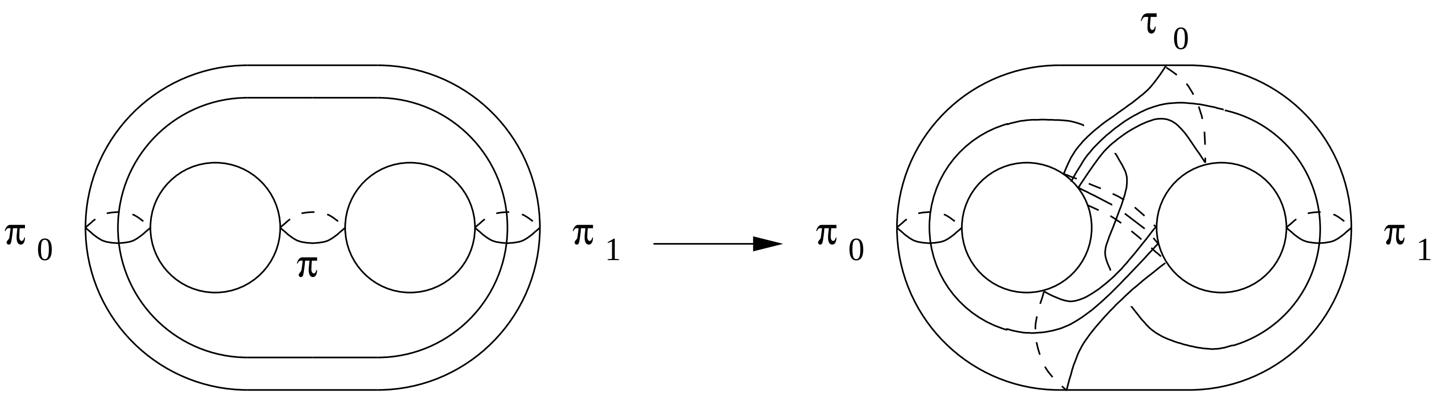
S. Cho's work on $\mathcal{G}$ (building on prior work of M. Scharlemann and E. Akbas) enables one to understand the action of the Goeritz group on $\mathcal{D}(H)$, and to work out the quotient $\mathcal{D}(H)/\mathcal{G}$:



Each of the vertices that is the image of a vertex of $\mathcal{D}(H)$ is a tunnel of some tunnel number 1 knot.

The combinatorial structure of $\mathcal{D}(H)/\mathcal{G}$ is reflected in the topology of the corresponding knot tunnels.



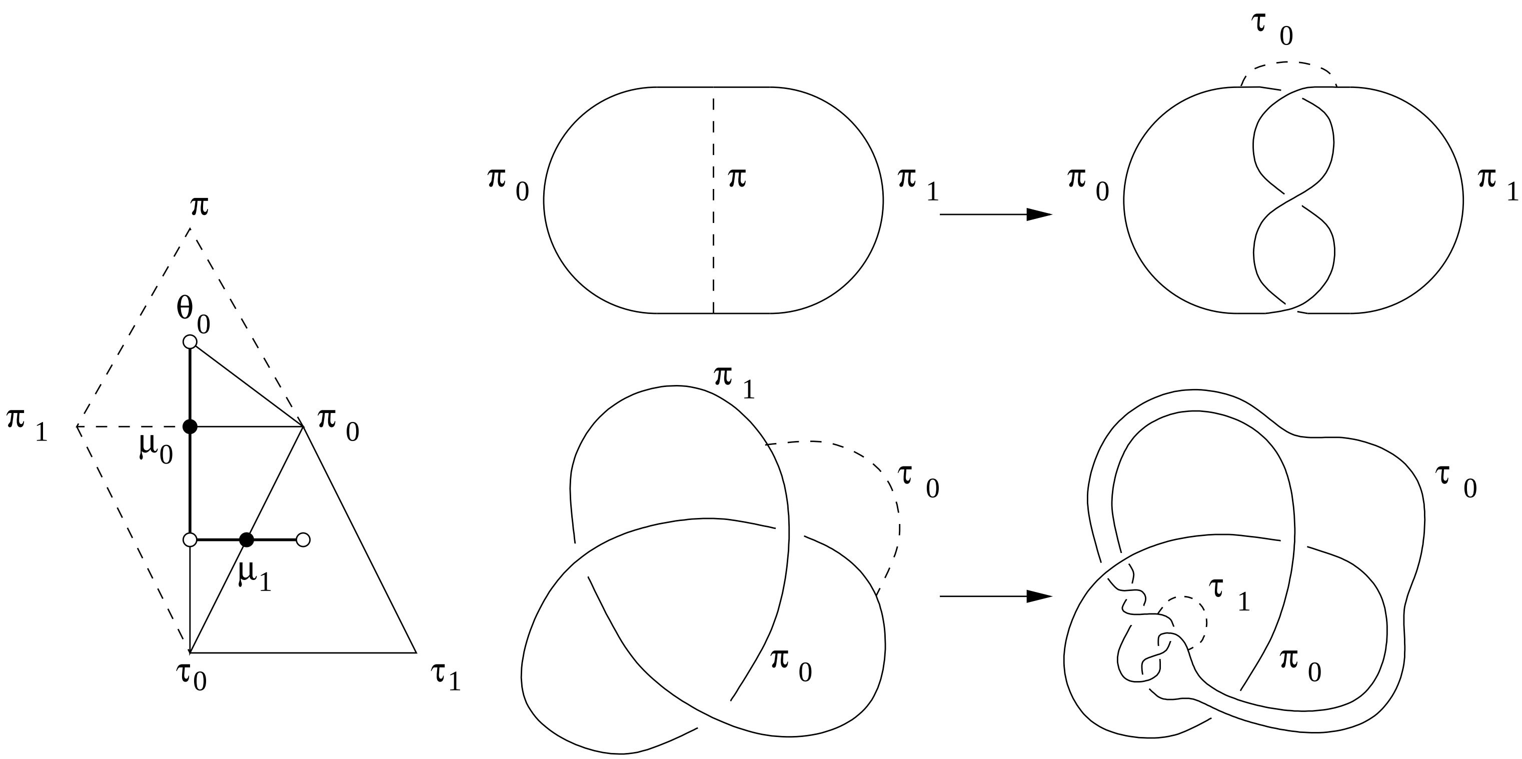


Here is an example. The triple θ_0 is the triple of standard disks $\{\pi_0, \pi_1, \pi\}$, and the complementary knots K_π, K_{π_0}, and K_{π_1} are trivial.

Removing π moves us to the vertex $\mu_0 = \{\pi_0, \pi_1\}$.

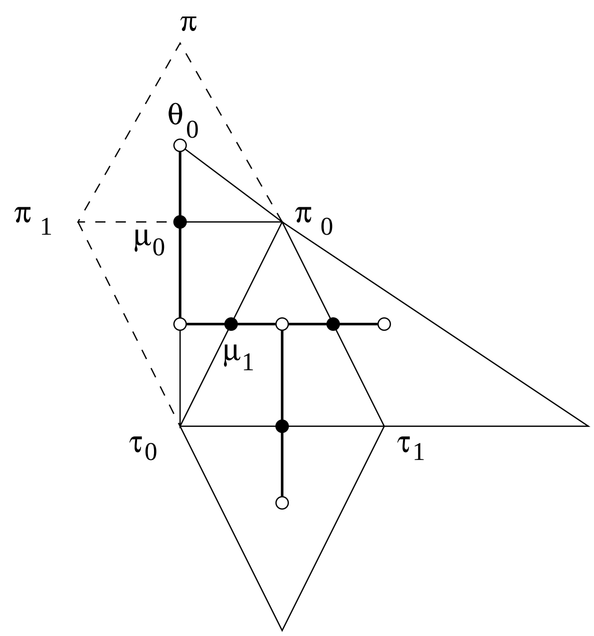
Adding τ_0 moves us to the vertex $\mu_0 \cup \{\tau_0\}$. The complementary knot K_{τ_0} is a trefoil and τ_0 represents its unique tunnel.

Continuing through the tree gives another step in this process:



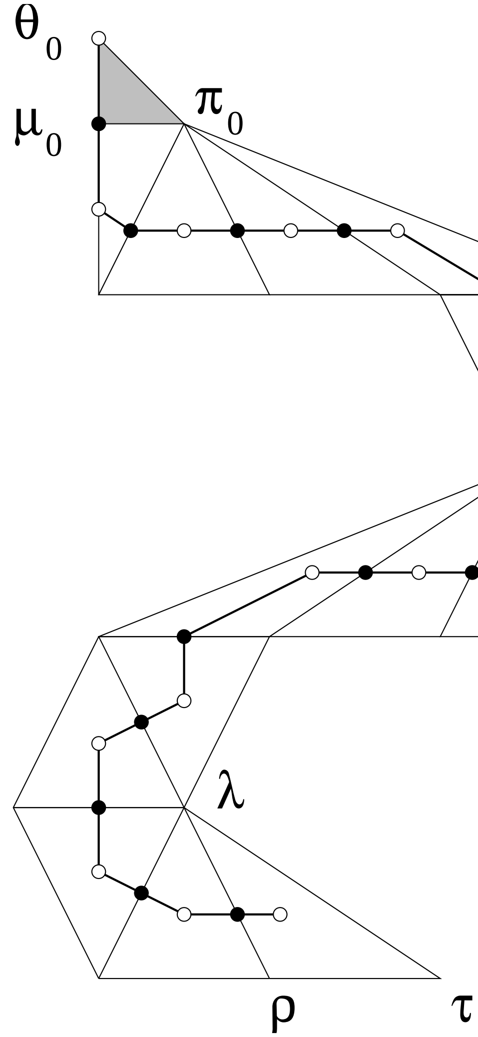
In short, a *cabling construction* is: Take one of the arcs of the knot and the tunnel arc, and attach the four ends using a rational tangle in a neighborhood of the other arc of the knot.

At the third and subsequent steps, the choice of which arc of the knot is kept and which is discarded affects the result. This is reflected in the fact that there are two ways to continue out of a white vertex:



Since $T/\mathcal{G}$ is a tree, every tunnel can be obtained by starting from the tunnel of the trivial knot and performing a *unique* sequence of cabling constructions.

The path in $T/\mathcal{G}$ that encodes this unique sequence of cablings is called the *principal path* of τ, shown here for a more complicated tunnel:

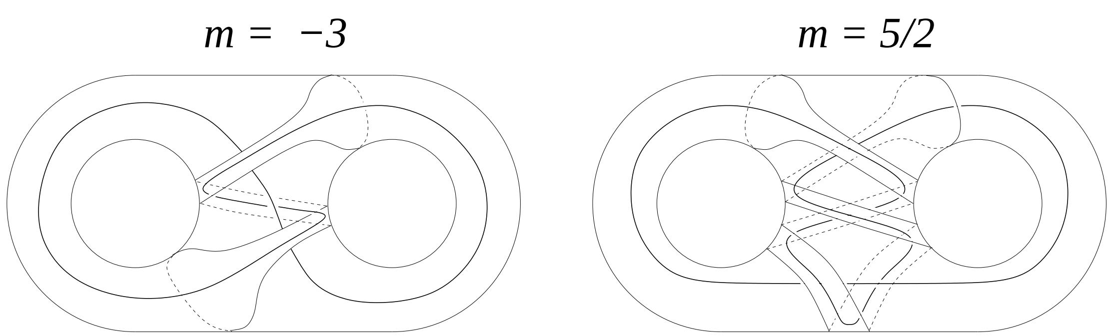


The last vertex $\{\lambda, \rho, \tau\}$ of the principal path is important, and is called the *principal vertex.*

A cabling operation is described by two items of information:

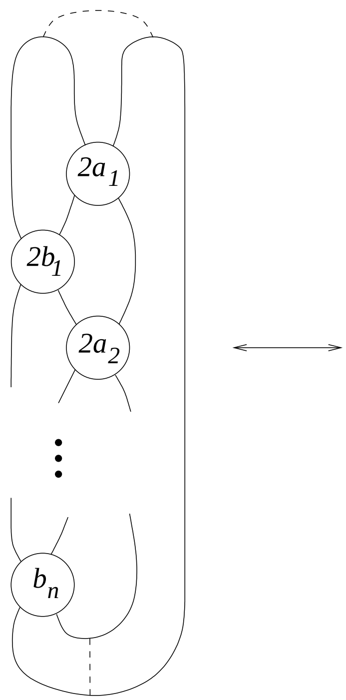
1. 1. A binary invariant s_i that tells which arc of K is kept and which is replaced by the rational tangle. These invariants are expressible in terms of the left-and-right turn sequence of the principal path.

2. 2. A rational "slope" parameter that tells which rational tangle to use.



The slope of the *final* cabling operation is (up to details of definition) the tunnel invariant discovered by M. Scharlemann and A. Thompson.

As an example, two-bridge knots are classified by a rational number (modulo $\mathbb{Z}$) whose reciprocal is given by the continued fraction with coefficients equal to the number of half-twists in the position shown here:

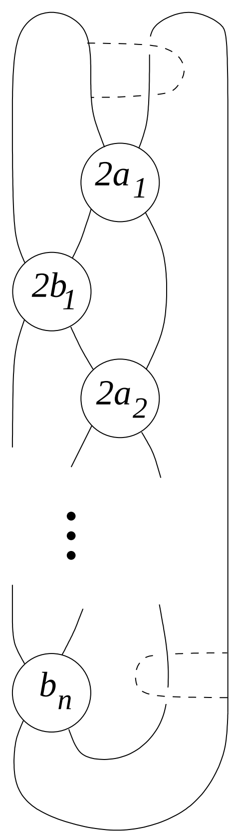


$$2a_1 + \cfrac{1}{2b_1 + \cfrac{1}{2a_2 + \cdots + \cfrac{1}{b_n}}}$$

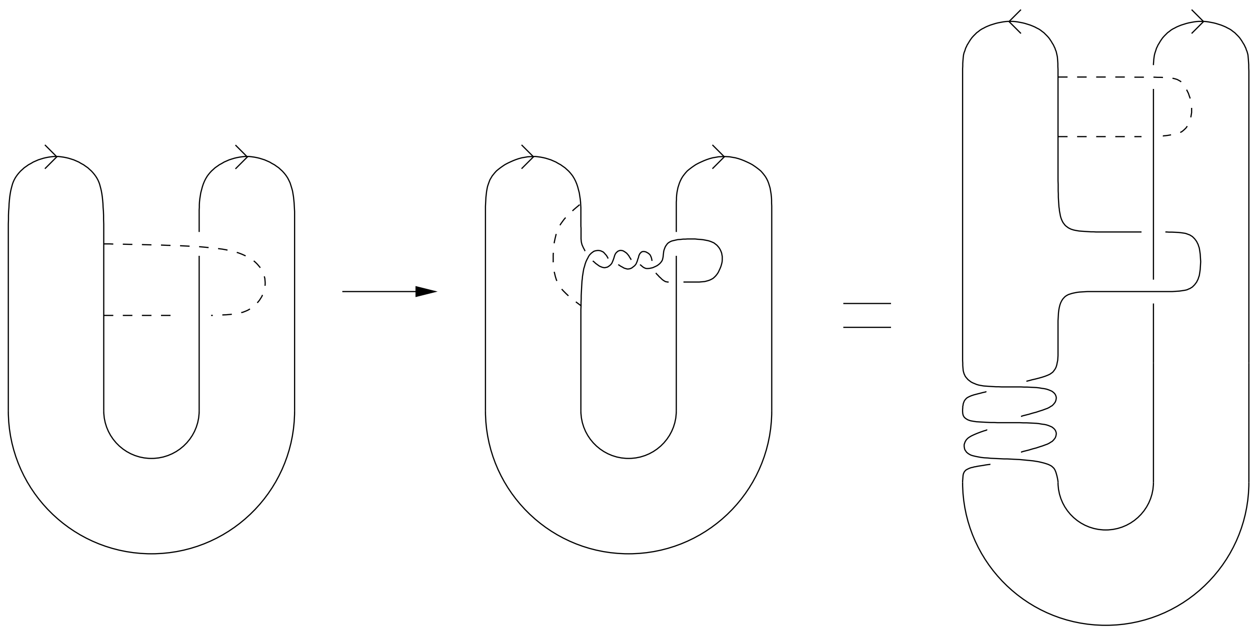
$$= [\, 2a_1 , 2b_1 , \ldots , 2a_n , b_n \,]$$

The tunnels shown here are called the "upper" or "lower" tunnels of the 2-bridge knot. They are the tunnels that are obtained from the trivial knot by a *single* cabling operation. For technical reasons, the first slope parameter is only well-defined in $\mathbb{Q}/\mathbb{Z}$, and not surprisingly it is essentially the standard invariant that classifies the 2-bridge knot.

The other tunnels of 2-bridge knots were classified by T. Kobayashi, K. Morimoto, and M. Sakuma. Besides the upper and lower tunnels, there are (at most) two other tunnels, shown here:



In the cabling sequence of these other tunnels, each cabling adds one full twist to the middle two strands, but an arbitrary number of half-twists to the left two strands:



These have slopes of the form $\pm 2 + \frac{1}{k}$, where k is related to the number of right-hand half-twists of the left two strands, and the calculation of the sign is complicated.

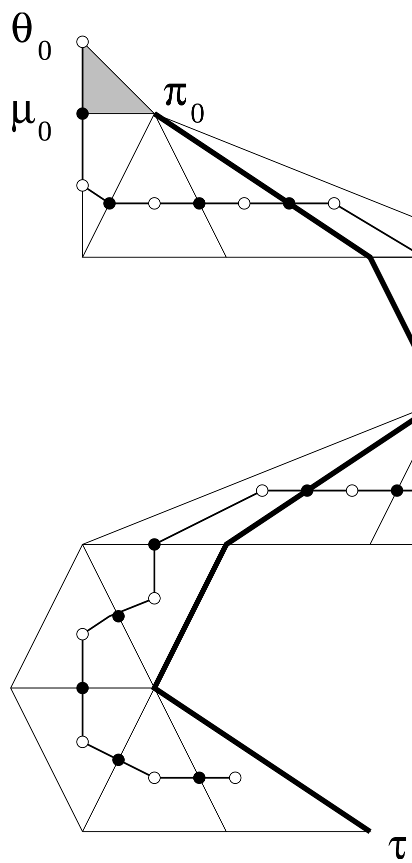
We wrote a program that takes the rational classifying invariant of the 2-bridge knot, and produces the slope parameters of the cablings in the cabling sequence of these other tunnels.

```
TwoBridge> slopes (33/19)
[ 1/3 ], 3, 5/3

TwoBridge> slopes (64793/31710)
[ 2/3 ], -3/2, 3, 3, 3, 3, 3, 7/3, 3, 3, 3, 3, 49/24

TwoBridge> slopes (3860981/2689048)
[ 13/27 ], 3, 3, 3, 5/3, 3, 7/3, 15/8, -5/3, -1, -3

TwoBridge> slopes (5272967/2616517)
[ 5/9 ], 11/5, 21/10, -23/11, -131/66
```

We also have calculated the invariants for all the tunnels of torus knots (Boileau-Rost-Zieschang and Moriah classified the tunnels of torus knots, for most cases there are three tunnels).

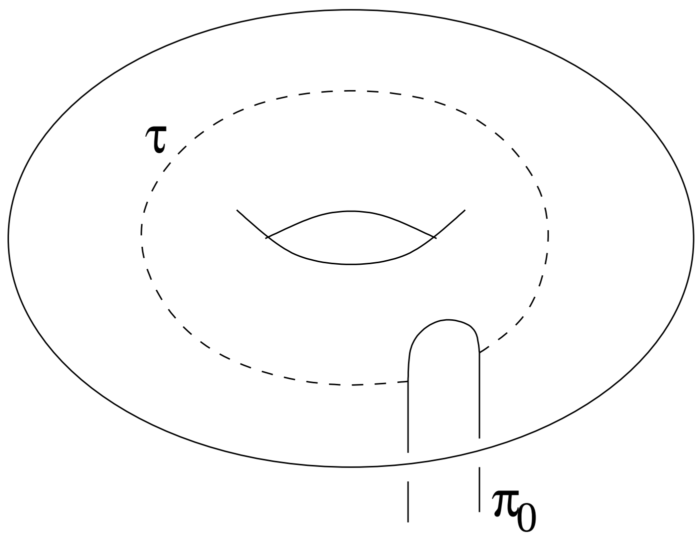
Some of the applications of our theory use a tunnel invariant called the *depth* of the tunnel. The depth of τ is the distance in the 1-skeleton of $D(H)/\mathcal{G}$ from the trivial tunnel π_0 to τ.

The tunnel that we saw earlier has depth 5:



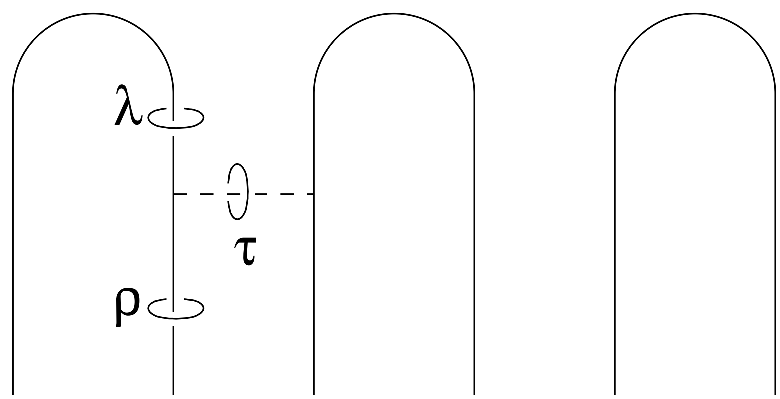
The depth-1 tunnels are exactly the type usually called $(1,1)$-tunnels.

Their associated knots can be put into 1-bridge position with respect to a torus $\times I$ (genus-1 1-bridge position). A $(1,1)$-tunnel for a $(1,1)$-knot looks like this with respect to some $(1,1)$-position:



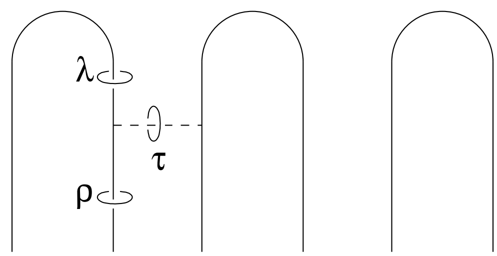
τ together with one of the arcs of the knot is an unknotted circle in S^3, so τ is disjoint from a trivial tunnel π_0, i. e. τ has depth 1. Conversely, it can be shown that every depth-1 tunnel is a $(1,1)$-tunnel.

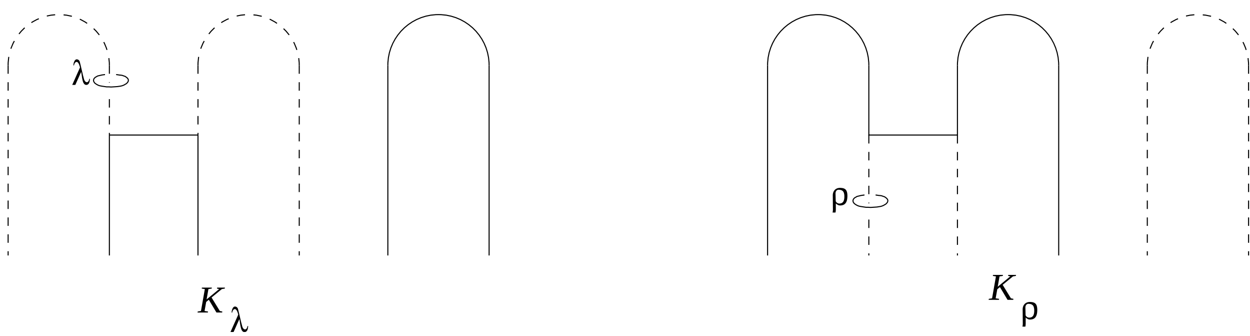
A powerful result about tunnels is the Tunnel Leveling Theorem of H. Goda-M. Scharlemann-A. Thompson. Roughly speaking, it says that a tunnel arc of a tunnel number 1 knot can be moved to lie in a level sphere of some minimal bridge position of the knot. Here is the picture, where $\{\lambda, \rho, \tau\}$ is the principal vertex of τ:



There is another configuration that only occurs for depth 1 tunnels, and for simplicity we will omit it from the discussion.

The knots whose tunnels are λ and ρ appear in this picture:





Thus

$$\mathrm{br}(K_\lambda) + \mathrm{br}(K_\rho) \le \mathrm{br}(K_\tau) \ ,$$

which was observed and used by Goda, Scharlemann, and Thompson.

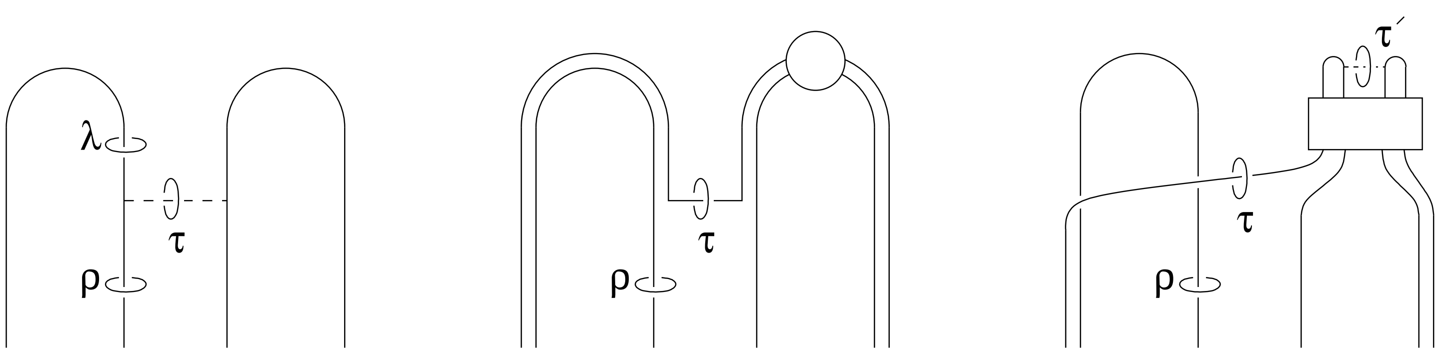
Using our cabling theory, we can prove the following *Tunnel Leveling Addendum:* When K_τ has depth at least 2,

$$\mathrm{br}(K_\lambda) + \mathrm{br}(K_\rho) = \mathrm{br}(K_\tau)\ .$$

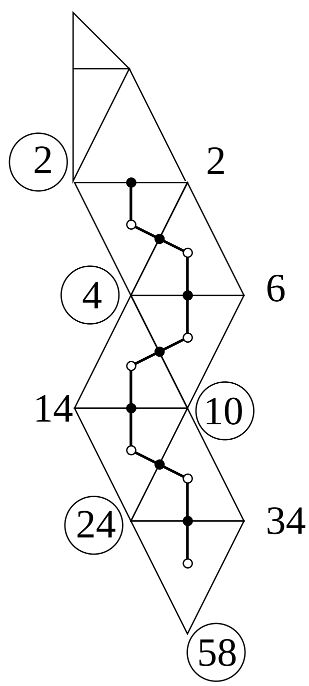
(When K_τ has depth 1, so that its principal vertex is $\{\pi_0, \rho, \tau\}$, the result is that

$$\mathrm{br}(K_\rho) \le \mathrm{br}(K_\tau) \le \mathrm{br}(K_\rho) + 1\ .)$$

The basic idea is that one can perform cabling so as to be "efficient" with respect to bridge number, as seen in the following picture:



Thus, for example, the "path of cheapest descent," i. e. the principal path for which the depth grows fastest relative to the bridge numbers, is:



From this one can easily work out the minimum bridge number of a tunnel of depth d.

Theorem 1 *For $d \geq 1$, the minimum bridge number of a knot having a tunnel of depth d is given recursively by a_d, where $a_1 = 2$, $a_2 = 4$, and $a_d = 2a_{d-1} + a_{d-2}$ for $d \geq 3$. Explicitly,*

$$a_d = \frac{(1+\sqrt{2})^d}{\sqrt{2}} - \frac{(1-\sqrt{2})^d}{\sqrt{2}}$$

and consequently $\lim_{d\to\infty} a_d - \frac{(1+\sqrt{2})^d}{\sqrt{2}} = 0$.

There is also a maximum bridge number theorem, in terms of the number of cablings:

Theorem 2 *Let $(F_1, F_2, \ldots)$ be the Fibonacci sequence $(1, 1, 2, 3, \ldots)$. The maximum bridge number of any tunnel number 1-knot having a tunnel produced by n cabling operations is F_{n+2}.*

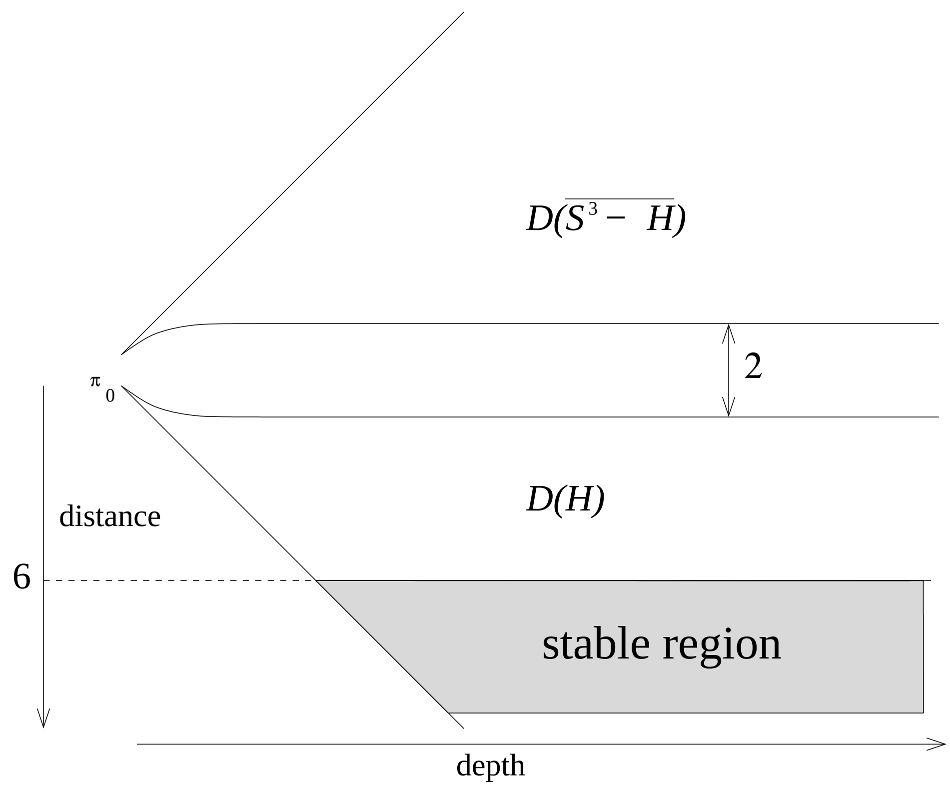
Another measure of complexity for a tunnel has been studied by J. Johnson, A. Thompson, Y. Minsky-Y. Moriah-S. Schleimer, and others:

The *(Hempel) distance* $\mathrm{dist}(\tau)$ is the distance in the *curve complex* $\mathcal{C}(\partial H)$ from $\partial\tau$ to a loop that bounds a disk in $\overline{S^3 - H}$.

Distance is related to depth by $\mathrm{dist}(\tau) - 1 \leq \mathrm{depth}(\tau)$. But depth is a finer invariant than distance:

The "middle" tunnels of torus knots all have Hempel distance 2, but their depths can be arbitrarily large (the depth of the middle tunnel of the (p, q)-torus knot is approximately the number of terms in the continued fraction expansion of p/q).

The disk complex imbeds in the curve complex $\mathcal{C}(\partial H)$, by taking each D to ∂D. Here is a schematic picture:



The "stable region" is the region of tunnels of distance at least 6. J. Johnson, using results of M. Scharlemann and M. Tomova, proved that

Theorem 3 *If K has a tunnel of distance at least 6, then this tunnel is the unique tunnel of K up to isotopy.*

A few words about knots of tunnel number ≥ 2, i. e. genus-(≥ 3) Heegaard splittings of knot exteriors:

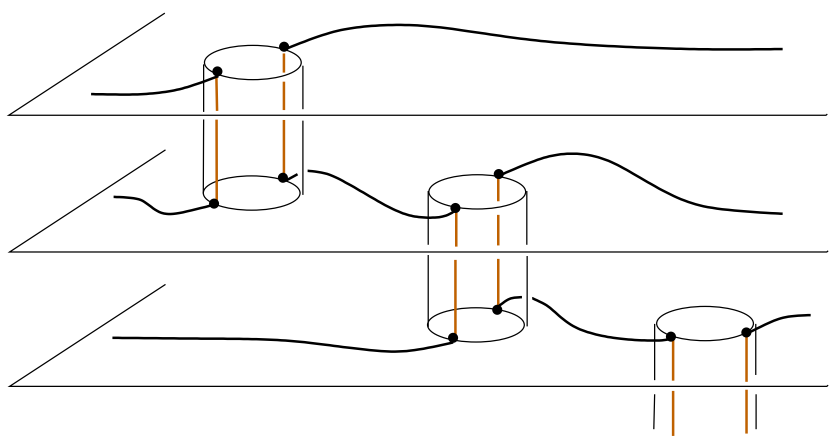
The analogous theory for knots of tunnel number larger than 1 would involve the disk complexes of higher genus handlebodies. For genus g, the disk complex is $(3g-4)$-dimensional (same as the curve complex). Although higher-genus disk complexes are contractible, their structures seem much more difficult to understand than for the genus-2 case.

For genus ≥ 3, it has not even been proven that the Goeritz group is finitely generated. A conjectural finite presentation has been given, and two proofs have been published, both incorrect.

A new application of complexes to knot theory: level position and arc distance.

A knot K in S^3 is said to be in genus-g 1-bridge position with respect to a genus-g Heegaard splitting $V \cup W$ of S^3 if each of $K \cap V$ and $K \cap W$ is a single arc that is parallel into the surface $F = \partial V = \partial W$.

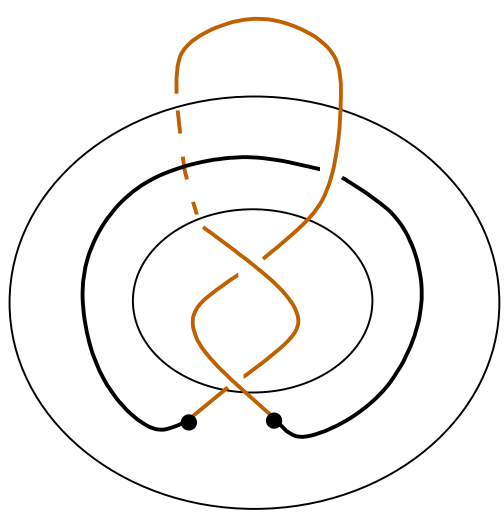
In a collar $F \times [0,1] \subset W$ of F in W, take n parallel copies of the form $F \times \{t\}$ and tube them together with $n-1$ unknotted tubes to obtain a surface G of genus gn in $F \times [0,1]$. We say that K lies in n-*level position* with respect to G if $K \subset G$, and moreover K meets each of the $n-1$ tubes in two arcs, each arc connecting the two ends of the tube.



Examples of level position appeared in work of M. Eudave-Muñoz, who used it to obtain closed incompressible surfaces in the complements of $(1,1)$-knots.

Every 1-bridge position of K is isotopic keeping $K \cap V$ in V and $K \cap W$ in W into some n-level position. The minimum such n is an invariant of the 1-bridge position, called the *level number.* Of course, for a knot having a genus-g 1-bridge position, the minimum level number over all 1-bridge positions is an invariant of the knot.

This figure shows that the torus level number of the figure-8 knot is at most 2, and hence equals 2 since the figure-8 knot is not a torus knot.



For a knot K in 1-bridge position, let $x, y \in F$ be the two points of $K \cap F$.

An arc σ_V in F from x to y is a *shadow* of $K \cap V$ if $K \cap V$ is isotopic in V, relative to $\{x, y\}$, to σ_V. A shadow σ_W of $K \cap W$ is defined similarly.

Each shadow is a point of the *arc complex* $\mathcal{A}(F)$ whose vertices are isotopy classes of arcs in F connecting x and y. It is known that $\mathcal{A}(F)$ is connected, so we can define an invariant of the 1-bridge position by

$$\text{dist}_F(K) = \min\{\text{dist}(\sigma_V, \sigma_W)\}$$

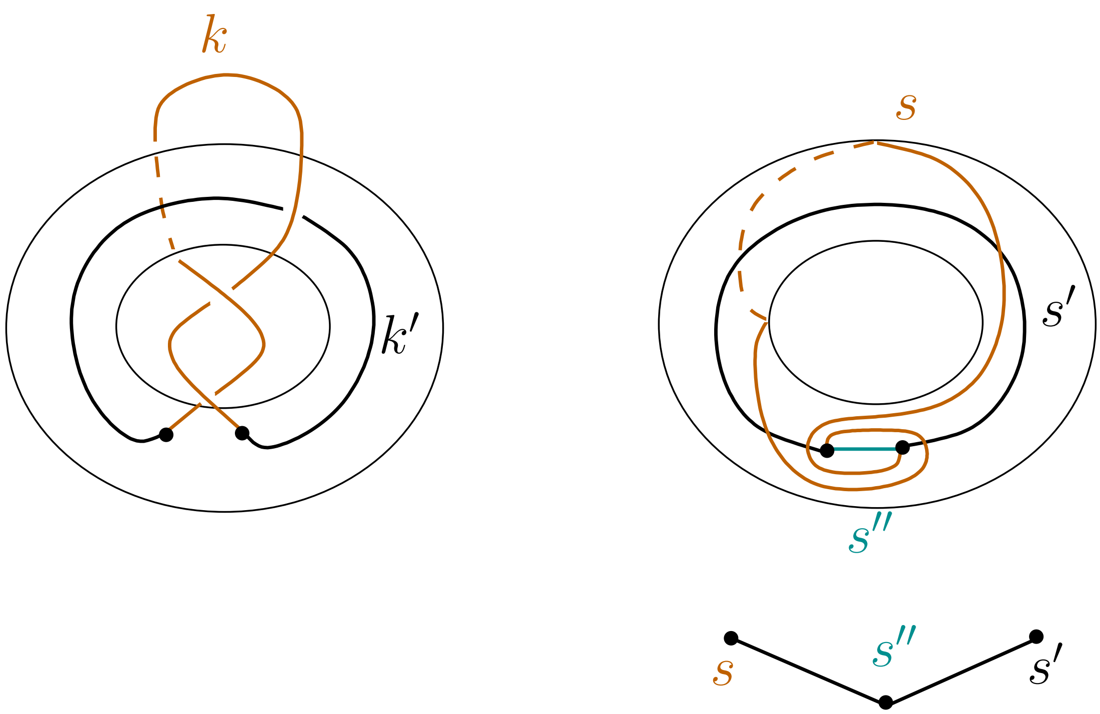
over all pairs of shadows of $K \cap V$ and $K \cap W$.

Of course, we can define the (genus-g) *arc distance* of K, denoted by dist(K), to be the minimum of $\text{dist}_F(K)$ over all genus-g 1-bridge positions of K.

The trivial knot is the only knot of distance 0.

A knot has torus distance 1 if and only if it is nontrivial torus knot.

This figure shows that the arc distance of the figure-8 knot is at most 2, and hence is 2 since the figure-8 knot is not a torus knot:



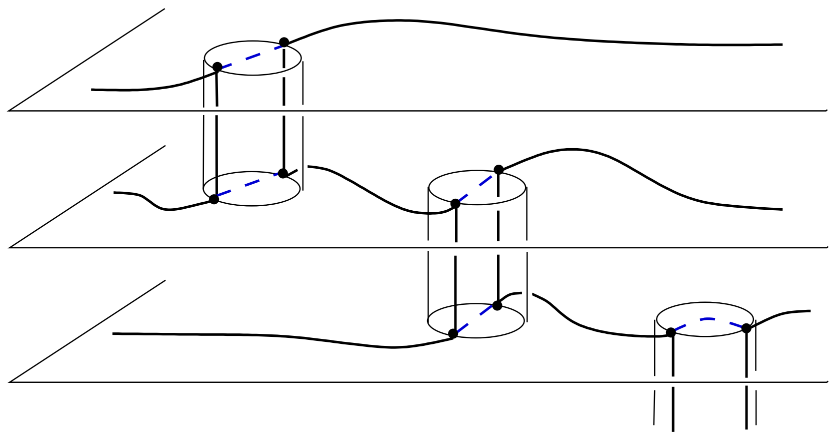
Theorem 4 *The level number of a* 1*-bridge position equals its arc distance.*

That is, K is isotopic (keeping $K \cap V$ in V and $K \cap W$ in W) into n-level position if and only if $\mathrm{dist}_F(K) \leq n$.

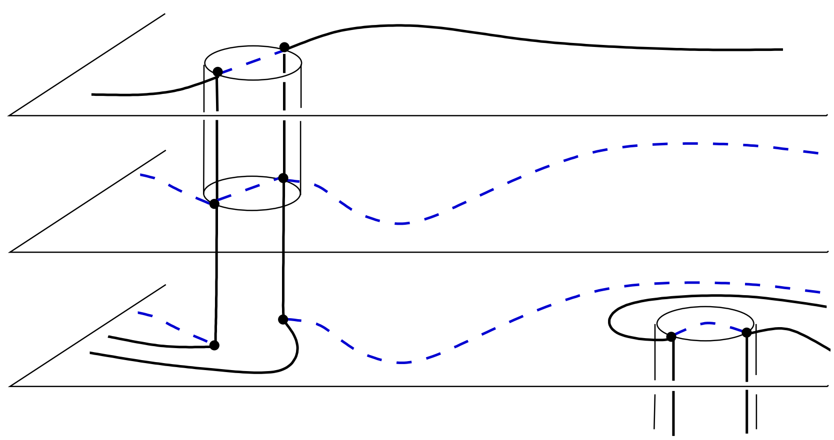
The theorem is not very hard to prove, but what is perhaps noteworthy is that it assigns a simple geometric interpretation to every possible distance.

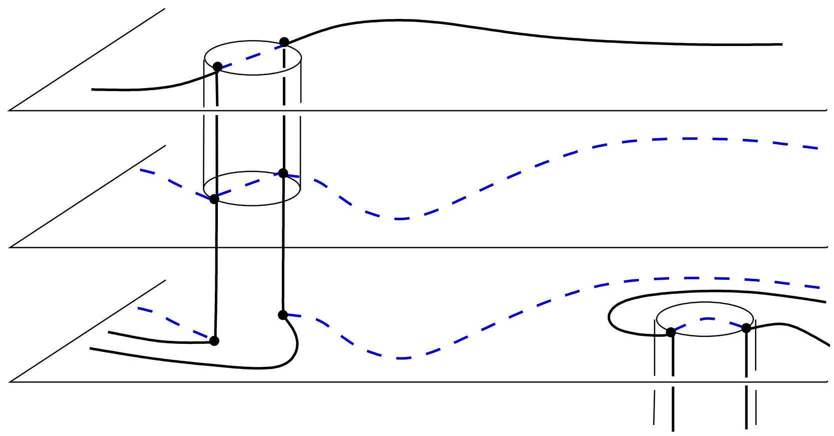
The idea of the proof:

Suppose K is in n-level position. Look at these dashed arcs drawn in the tubes.



Move the knot by an isotopy that shrinks its two arcs in the second level.





The second dotted arc is stretched out to an arc with endpoints $\{x, y\}$, and disjoint from the first dotted arc.

Repeat this process on each intermediate level, ending up with $n-1$ arcs which form a path between two shadows. So $\mathrm{dist}_F(K) \leq n$.

The other direction— a path of length n in the arc complex gives an n-level position— is essentially a matter of reversing this process.

There is probably more to be done with this idea. Like disk complexes, arc complexes tend to be easier to understand than curve complexes.

We are currently working on $(1,1)$-knots by regarding them as braids of two points in the torus, a viewpoint already used by Choi and Ko. In work in progress, we have methods to calculate the cabling slope invariants and the level number in terms of a braid word that describes the $(1,1)$-knot.