The tunnel leveling addendum

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Geometric Topology in 3 and 4 Dimensions

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(joint work with Sangbum Cho)

Today's topics:

1. The tree of knot tunnels: a quick review.

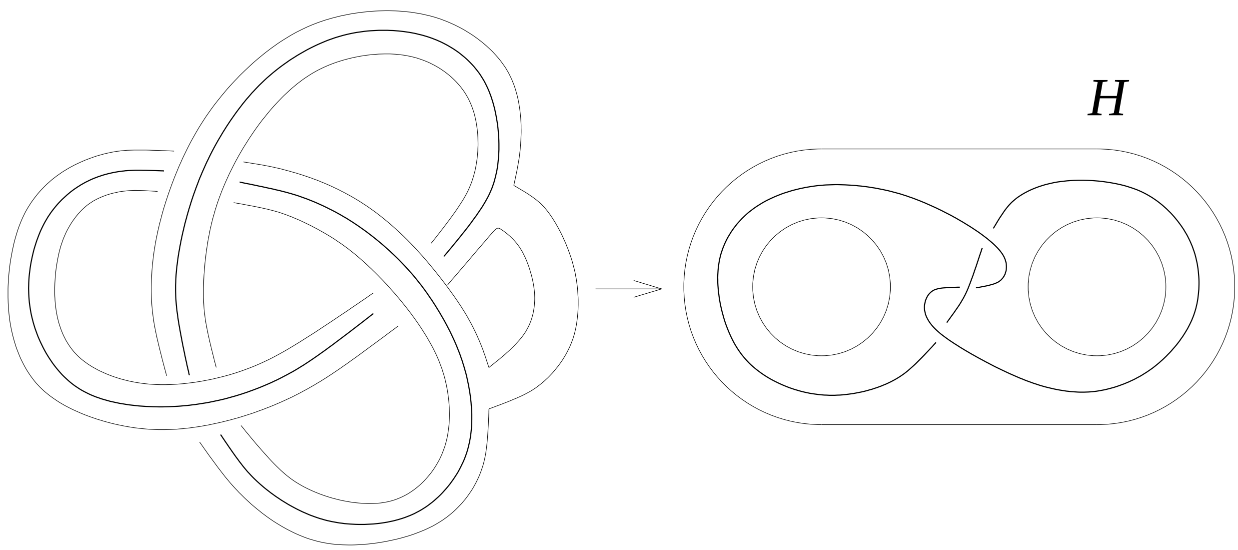
2. The Tunnel Leveling Addendum and some of its applications.

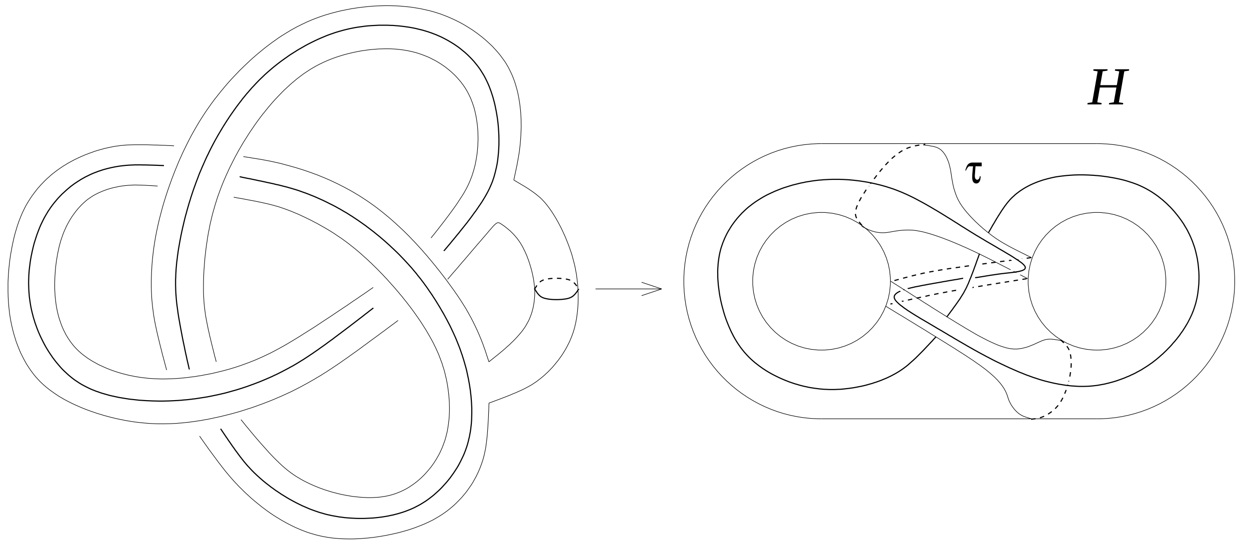
3. Recent work on $(1,1)$-tunnels.

4. Work in progress on knots with more than one equivalence class of tunnels.

Our study of tunnel number 1 knots was originally motivated by some work of M. Scharlemann and A. Thompson, who defined a rational invariant of a knot tunnel.

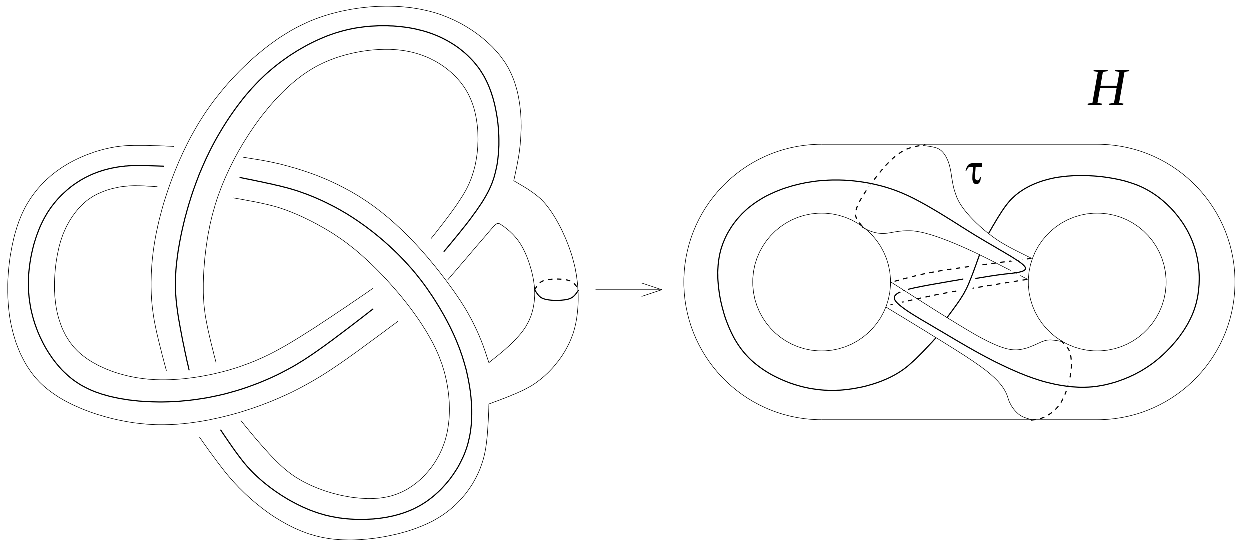
Let's recall that a tunnel of a tunnel number 1 knot corresponds to a genus-2 Heegaard splitting of the knot exterior:





Under an isotopy moving the neighborhood of the knot and the tunnel to the standard handlebody H, the cocore disk of the tunnel moves to a disk τ in H, and the knot moves to a core of the complementary solid torus of τ in H.

τ is well-defined up to a homeomorphism of H that results from moving H by isotopy through S^3 and back to its standard position. The group of isotopy classes of such homeomorphisms of H is called the *Goeritz group* $\mathcal{G}$.

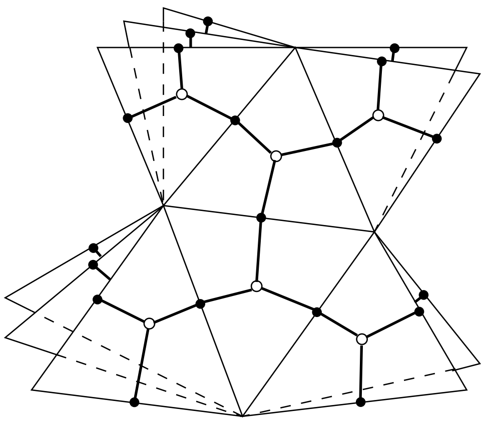


On the other hand, for each nonseparating disk τ in H, the core of its complementary solid torus is a knot K_τ, and τ is the cocore disk of a tunnel of K_τ.

Summary: *The collection of all tunnels of all tunnel number* 1 *knots corresponds to the collection of nonseparating disks in the standard handlebody* H *in* S^3*, modulo the action of* $\mathcal{G}$*.*

To understand the disks in H, we examine the complex $\mathcal{D}(H)$ of nonseparating disks in H, which is analogous to the curve complex of a surface.

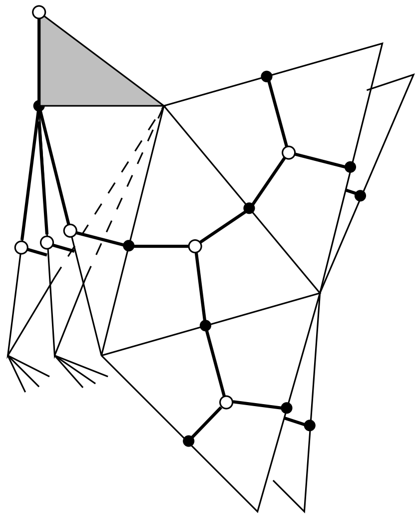
$\mathcal{D}(H)$ looks like this, with countably many 2-simplices meeting at each edge:



and it deformation retracts to the tree T shown in this figure.

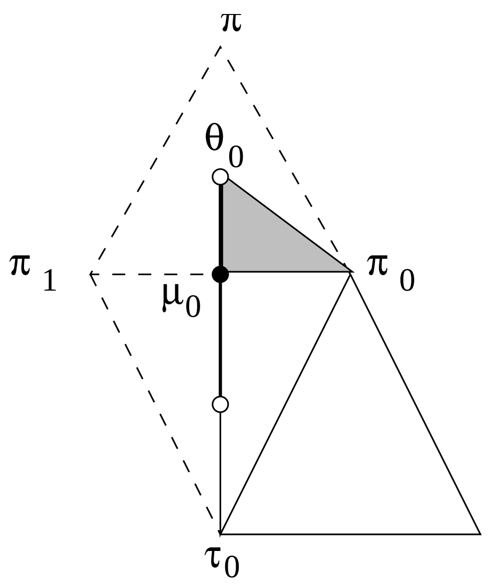
Each white vertex of T is a triple of nonseparating disks, and each black vertex is a pair.

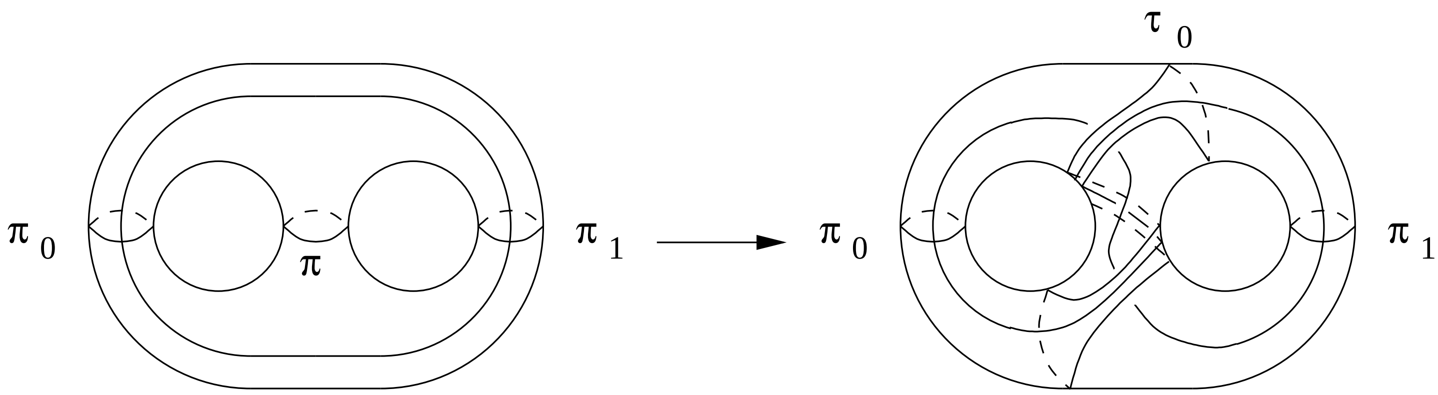
Using S. Cho's work on $\mathcal{G}$ (which builds on prior work of M. Scharlemann and E. Akbas), we can understand the action of $\mathcal{G}$ on $\mathcal{D}(H)$, and work out the quotient $\mathcal{D}(H)/\mathcal{G}$:



Each of the vertices that is the image of a vertex of $\mathcal{D}(H)$ is a tunnel of some tunnel number 1 knot.

The combinatorial structure of $\mathcal{D}(H)/\mathcal{G}$ is reflected in the topology of the corresponding knot tunnels.



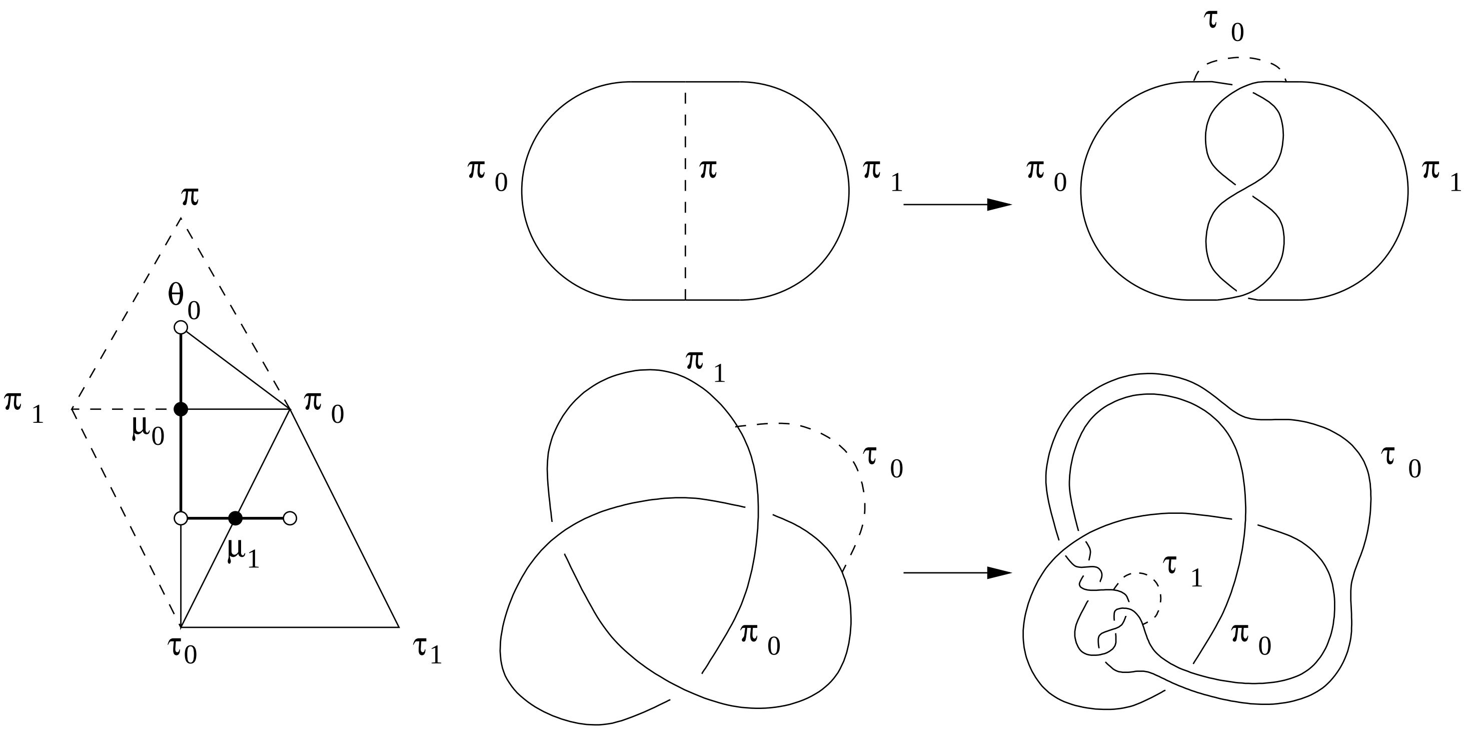


Here is an example. The white vertex θ_0 is the triple of standard disks $\{\pi_0, \pi_1, \pi\}$, and the complementary knots K_π, K_{π_0}, and K_{π_1} are trivial.

Removing π moves us to the black vertex $\mu_0 = \{\pi_0, \pi_1\}$.

Adding τ_0 moves us to the white vertex $\mu_0 \cup \{\tau_0\}$. The complementary knot K_{τ_0} is a trefoil and τ_0 represents its unique tunnel.

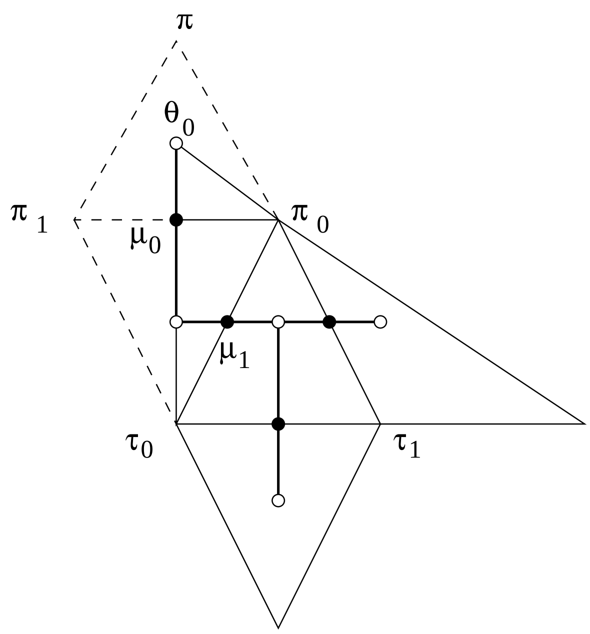
Continuing through the tree gives another step in this process:



In short, a *cabling construction* is: Take the tunnel arc and one of the arcs of the knot, and attach the four ends using a rational tangle in a neighborhood of the other arc of the knot.

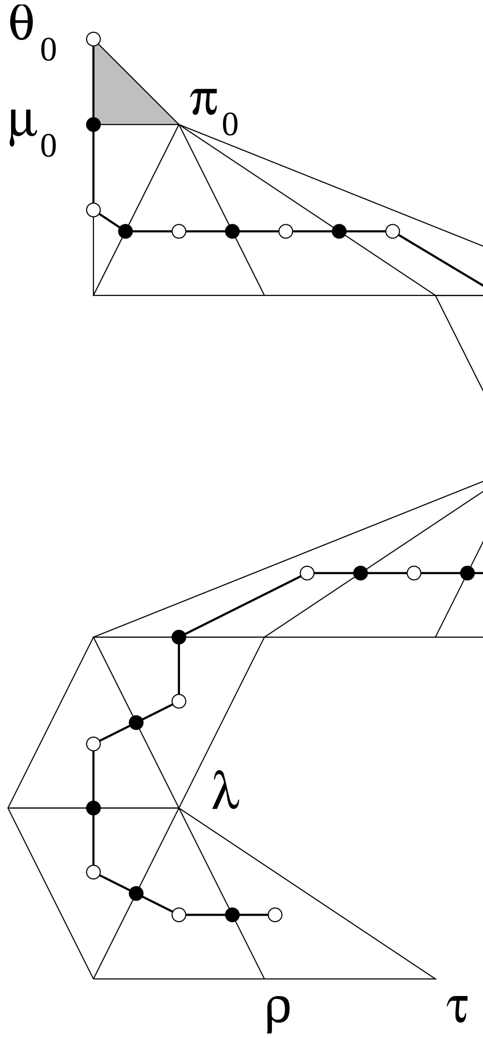
This produces a new tunnel number 1 knot, and a well-defined tunnel of it.

At the third and subsequent steps, the choice of which arc of the knot is kept and which is replaced affects the result. This corresponds to the fact that there are two ways to continue out of a white vertex:



Since $T/\mathcal{G}$ is a tree, every tunnel can be obtained by starting from the tunnel of the trivial knot and performing a *unique* sequence of cabling constructions.

The path in $T/\mathcal{G}$ that encodes this unique sequence of cablings is called the *principal path* of τ, shown here for a more complicated tunnel:

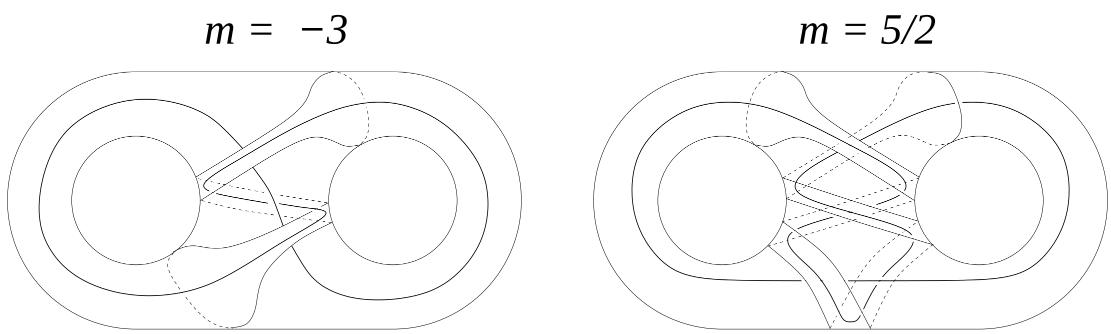


The last vertex $\{\lambda, \rho, \tau\}$ of the principal path is important, and is called the *principal vertex.*

A cabling operation is described by two items of information:

1. A binary invariant s_i that tells which arc of K is kept and which is replaced by the rational tangle. These invariants are expressible in terms of the left-and-right turn sequence of the principal path.

2. A rational "slope" parameter that tells which rational tangle to use.

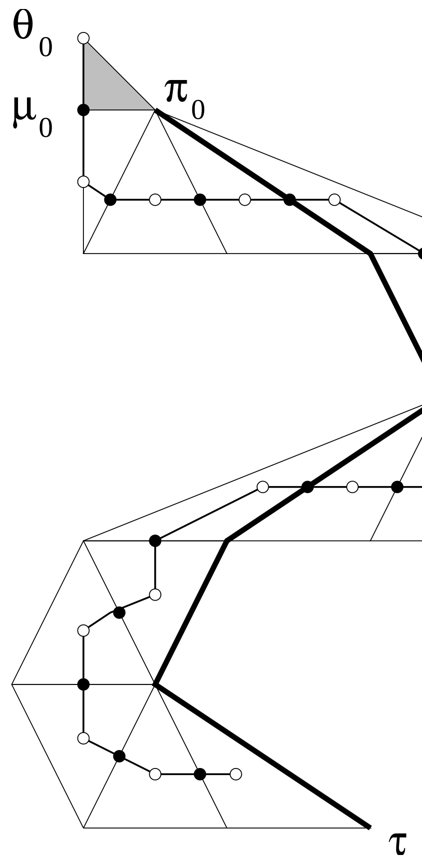


The slope of the *final* cabling operation is (up to details of definition) the tunnel invariant discovered by M. Scharlemann and A. Thompson.

We have calculated the sequences of slope invariants for all tunnels of two-bridge knots and torus knots.

The *depth* of a tunnel is the distance in the 1-skeleton of $D(H)/\mathcal{G}$ from the trivial tunnel π_0 to τ.

The tunnel that we saw earlier has depth 5:

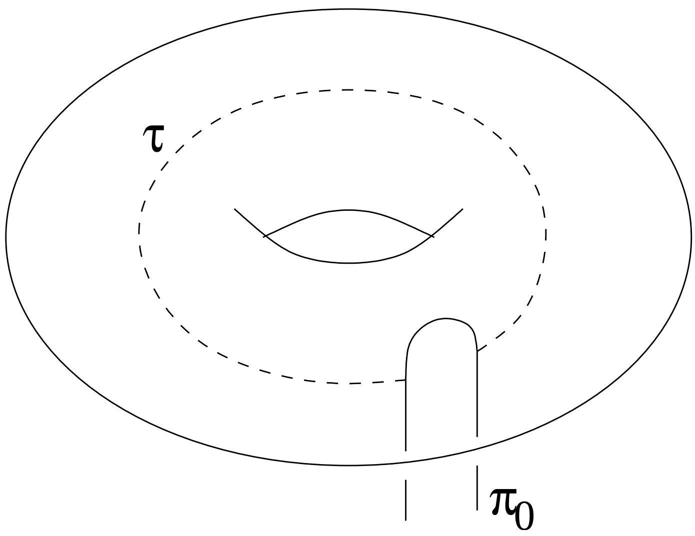


Depth is related to the Heegaard distance of the associated genus-2 Heegaard splitting of the knot exterior:

$$\text{distance} - 1 \leq \text{depth} \ .$$

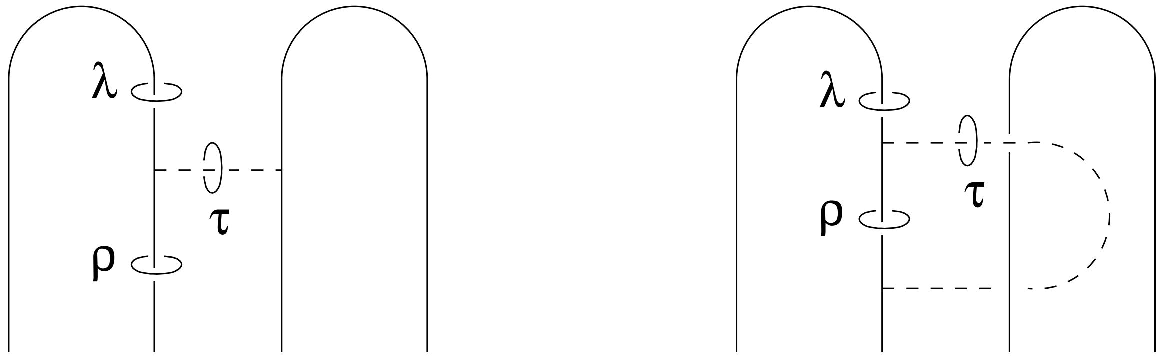
The depth-1 tunnels are exactly the type usually called $(1, 1)$-tunnels.

Their associated knots can be put into 1-bridge position with respect to a Heegaard torus in S^3. A $(1, 1)$-tunnel for a $(1, 1)$-knot looks like this with respect to some $(1, 1)$-position:



τ together with one of the arcs of the knot is an unknotted circle in S^3, so τ is disjoint from a trivial tunnel π_0, i. e. τ has depth 1. Conversely, it can be shown that every depth-1 tunnel is a $(1, 1)$-tunnel.

A powerful result about tunnels is the Tunnel Leveling Theorem of H. Goda-M. Scharlemann-A. Thompson. Roughly speaking, it says that a tunnel arc of a tunnel number 1 knot can be moved to lie in a level sphere of some minimal bridge position of the knot. There are two cases:

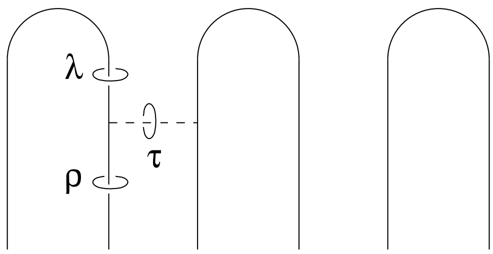


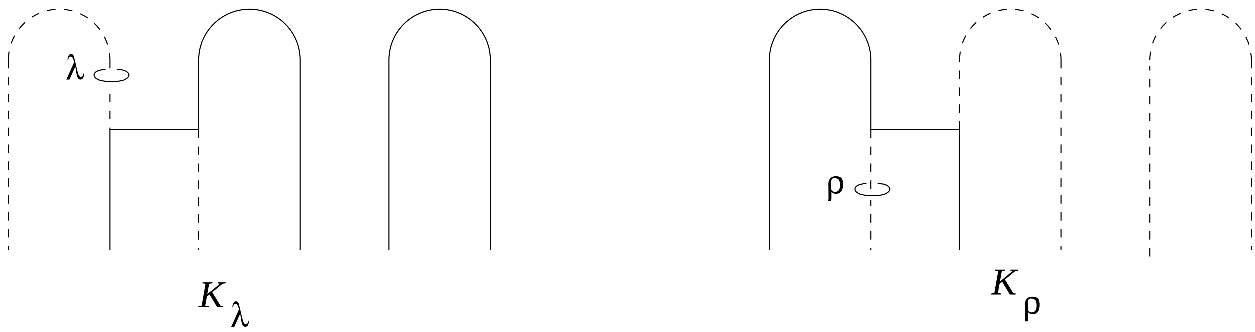
The first is a level arc, and the second is an "eyeglass," which can occur only when τ is a $(1,1)$-tunnel.

In other work, Scharlemann and Thompson showed that the disks λ and ρ in these pictures, together with τ, form the principal vertex $\{\lambda, \rho, \tau\}$ of τ.

In the eyeglass case, λ is the trivial tunnel, i. e. the principal vertex of τ is $\{\pi_0, \rho, \tau\}$.

The knots K_λ and K_ρ appear in this picture:





Thus

$$\mathrm{br}(K_\lambda) + \mathrm{br}(K_\rho) \leq \mathrm{br}(K_\tau) \ ,$$

which was observed and used by Goda, Scharlemann, and Thompson.

Using our theory, we can prove the following *Tunnel Leveling Addendum:*

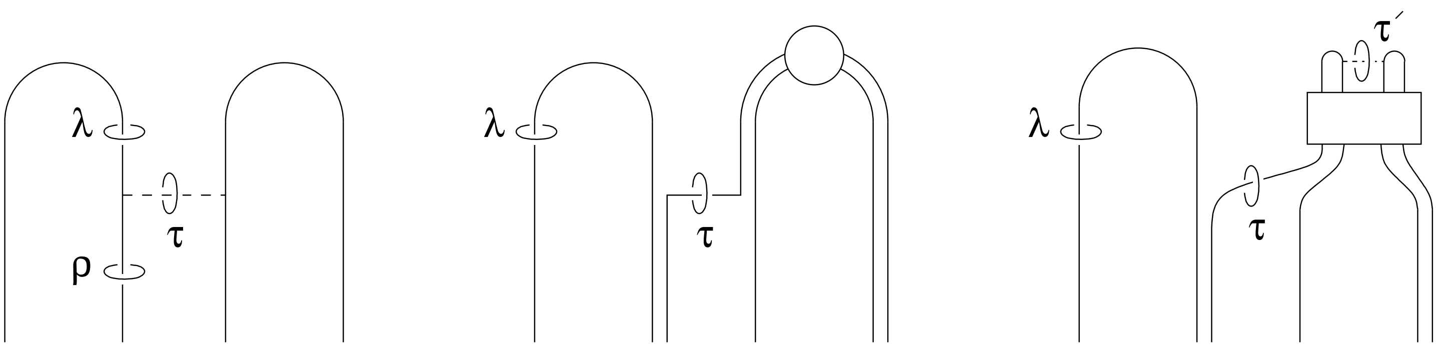
1. When K_τ has depth at least 2,

$$\mathrm{br}(K_\lambda) + \mathrm{br}(K_\rho) = \mathrm{br}(K_\tau)\ .$$

2. When K_τ has depth 1, with principal vertex $\{\pi_0, \rho, \tau\}$,

$$\mathrm{br}(K_\rho) \leq \mathrm{br}(K_\tau) \leq \mathrm{br}(K_\rho) + 1\ .$$

The basic idea is that one can perform cabling inductively so as to be "efficient" with respect to bridge number, as seen in the following picture:



There are similar but slightly more complicated configurations for eyeglass tunnels.

A careful inductive argument achieves a more precise statement:

Tunnel Leveling Addendum: Let τ be a tunnel with principal vertex $\{\lambda, \rho, \tau\}$. If τ is depth 1, write its principal vertex as $\{\pi_0, \rho, \tau\}$. Assume that τ is not the tunnel of the trivial knot or a $(2n+1, 2)$ torus knot. Then either

(a) All level positions of τ are level arc positions, and $\mathrm{br}(K_\tau) = \mathrm{br}(K_\rho) + \mathrm{br}(K_\lambda)$, or

(b) All level positions of τ are eyeglass positions, τ has depth 1, and $\mathrm{br}(K_\tau) = \mathrm{br}(K_\rho)$.

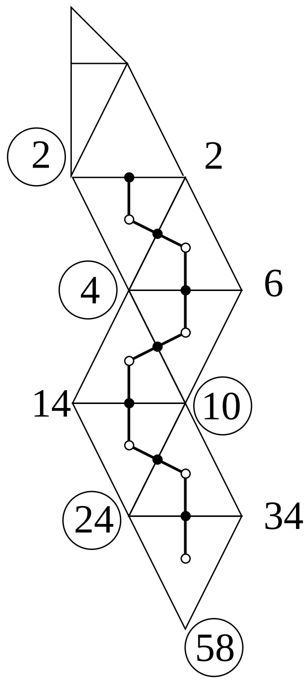
Corollary: When τ has depth ≥ 2,

$$\mathrm{br}(K_\tau) = \mathrm{br}(K_\rho) + \mathrm{br}(K_\lambda) \ ,$$

and when τ has depth 1,

$$\mathrm{br}(K_\rho) \leq \mathrm{br}(K_\tau) \leq \mathrm{br}(K_\rho) + 1 \ .$$

Thus, for example, the "path of cheapest descent," i. e. the principal path for which the depth grows fastest relative to the bridge numbers, is:



The figure shows that the smallest bridge number for a knot with a tunnel of depth 5 is 58.

One can easily work out a recursion that tells the minimum bridge number of a tunnel of depth d.

Theorem 1 *For $d \geq 1$, the minimum bridge number of a knot having a tunnel of depth d is given recursively by a_d, where $a_1 = 2$, $a_2 = 4$, and $a_d = 2a_{d-1} + a_{d-2}$ for $d \geq 3$. Explicitly,*

$$a_d = \frac{(1+\sqrt{2})^d}{\sqrt{2}} - \frac{(1-\sqrt{2})^d}{\sqrt{2}}$$

and consequently $\lim_{d\to\infty} a_d - \frac{(1+\sqrt{2})^d}{\sqrt{2}} = 0$.

There is also a maximum bridge number theorem, in terms of the number of cablings:

Theorem 2 *Let $(F_1, F_2, \ldots)$ be the Fibonacci sequence $(1, 1, 2, 3, \ldots)$. The maximum bridge number of any tunnel number 1-knot having a tunnel produced by n cabling operations is F_{n+2}.*

To try to understand $(1,1)$-tunnels better, we have developed a method for calculating the slope invariants of any $(1,1)$-tunnel in terms of a description of the $(1,1)$-position as a braid of two points in the torus.

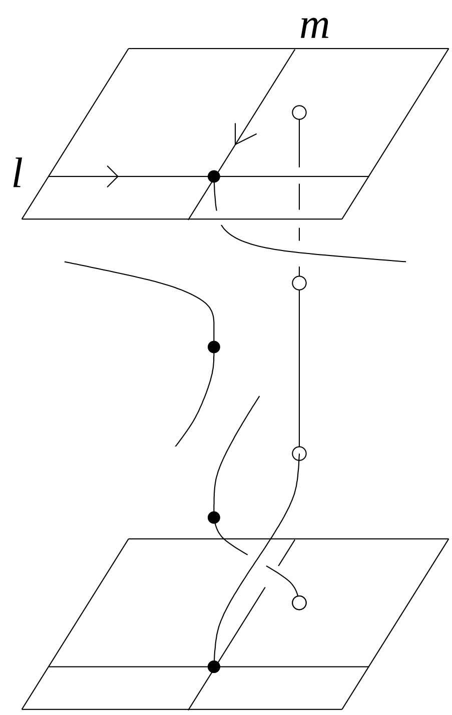
The braid group of two points “black” and “white” in the torus is generated by three simple braids:

δ_ℓ - push black point around longitude

δ_m - push black point around meridian

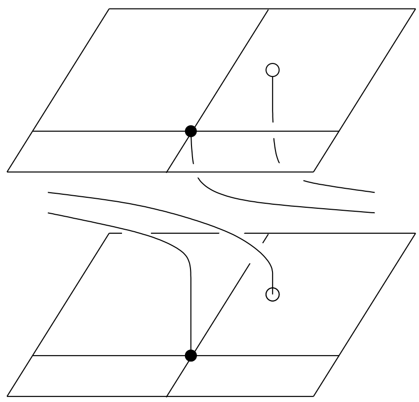
σ - half-twist interchanging black and white points

The braid $\delta_\ell \delta_m \sigma$ is shown here:



Putting trivial arcs at the top and bottom of a braid gives a knot in $(1,1)$-position.

A sequence $\sigma\delta_\ell\sigma\delta_\ell$, anywhere in the braid, does not change the knot, up to $(1,1)$-isotopy:

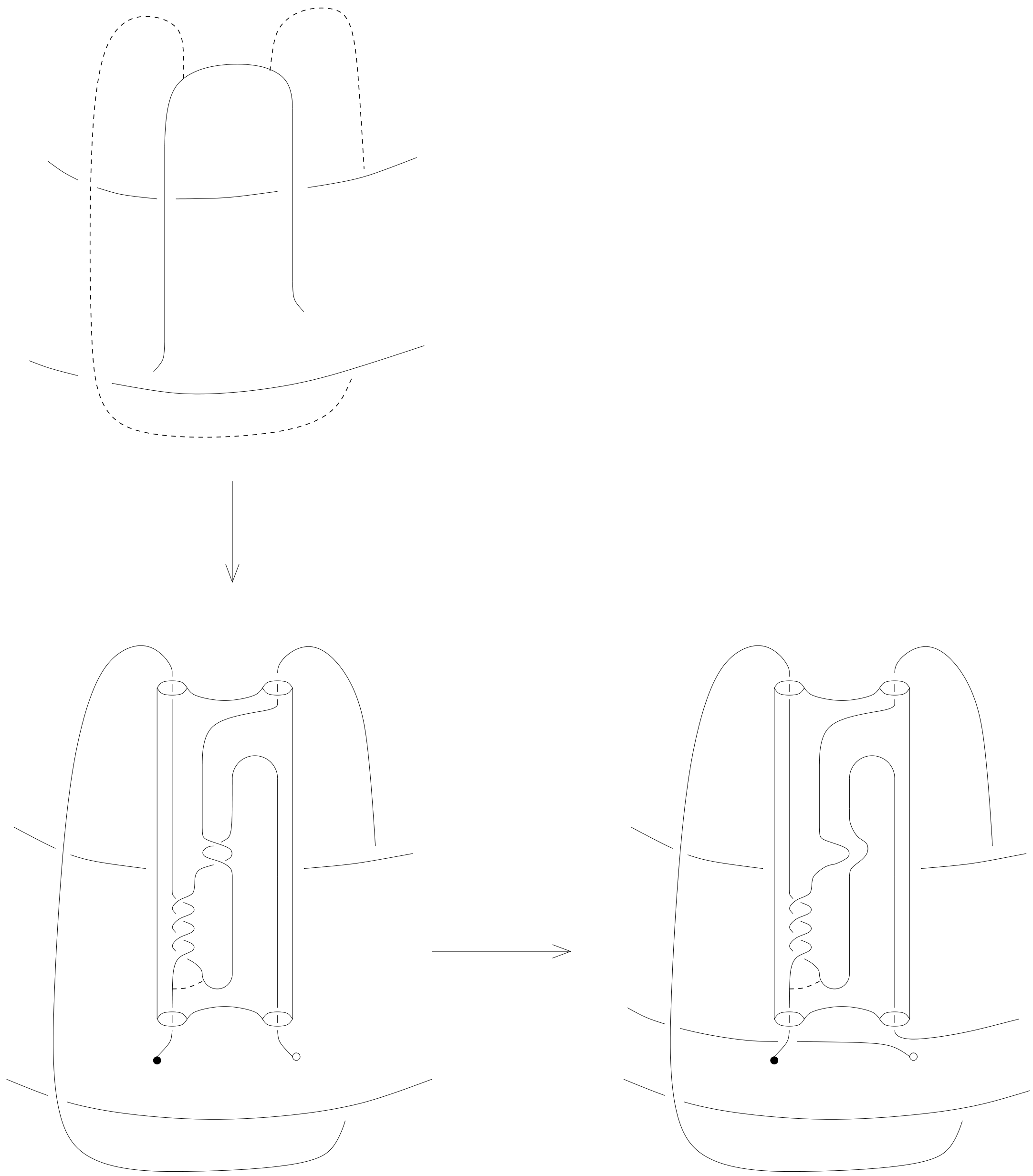


Similarly, a sequence $\sigma\delta_m\sigma\delta_m$ has no effect. Quotienting out Birman's presentation of the braid group by the subgroup generated by these two elements (which is the center of the braid group), gives the (reduced) braid group:

$$\mathcal{B} = \langle\ \delta_m, \delta_\ell, \sigma \mid (\sigma\delta_m)^2 = (\sigma\delta_\ell)^2 = 1, \\ \delta_\ell^{-1}\delta_m\delta_\ell\delta_m^{-1} = \sigma^2\ \rangle$$

Each word in $\mathcal{B}$ determines a knot in $(1,1)$-position (many words give the same knot in an isotopic $(1,1)$-position).

We can understand how a cabling operation that takes a $(1,1)$-tunnel and gives another changes the braid word. Here is a picture of the "unwinding" of a cabling construction:



Using a continued fraction method, we can compute the braid word in terms of the slopes of the cablings.

Conversely, we can take a braid word, rewrite it in a standard from, and read off the cabling slopes. Thus we achieve an effective translation between braid words and cabling sequences for $(1,1)$-tunnels.

Putting this on a computer, we can do a number of computations.

For example, we give a braid word such as

$$\delta_m^3 \sigma^{-2} \delta_\ell^4 \sigma^{-4} \delta_m^{-1} \sigma^{-4} \delta_\ell^3$$

and obtain the slope sequences for the upper and lower $(1,1)$-tunnels:

```
> upperSlopes( 'm 3 s -2 l 4 s -4 m -1 s -4 l 3' )
[ 21/25 ], 443/78, -15, -15

> lowerSlopes( 'm 3 s -2 l 4 s -4 m -1 s -4 l 3' )
[ 16/19 ], -7, -7, -7, -195/31, -5, -5
```

Conversely, we can give a slope sequence such as

$$[21/25], 443/78, -15, -15$$

and recover a braid word for the knot:

```
> print braidWord( [21,25,443,78,-15,1,-15,1] )
m 3 s -3 m -1 l -3 m 1 l -1 s -4 m 1 s -4 m -1 l -2 m 1
l -1

> upperSlopes( 'm 3 s -3 m -1 l -3 m 1 l -1 s -4
m 1 s -4 m -1 l -2 m 1 l -1' )
[ 21/25 ], 443/78, -15, -15
```

Stringing these together gives a direct calculation of the slopes of one tunnel associated to a $(1, 1)$-position from the slopes of the other:

```
> dualSlopes( [21,25,443,78,-15,1,-15,1] )
[ 16/19 ], -7, -7, -7, -195/31, -5, -5

> dualSlopes( [16,19,-7,1,-7,1,-7,1,-195,31,-5,1,-5,1] )
[ 21/25 ], 443/78, -15, -15
```

We achieve confirmation of the original calculations of slope sequences for tunnels of 2-bridge and $(1,1)$-tunnels of torus knots. For example, for 2-bridge knots we have

```
> twoBridge( 413, 227 )
Upper simple tunnel:  [ 131/413 ]
Upper semisimple tunnel:  [ 1/3 ], 15/7, 9/5
Lower simple tunnel:  [ 227/413 ]
Lower semisimple tunnel:  [ 2/5 ], -1, -3/2, 1, 1, 1, 3

> print upperSemisimpleBraidWord( 413, 227 )
m -1 s -6 m -1 s 6 m -1 s 1 l -1
> print lowerSimpleBraidWord( 413, 227 )
m -1 s 1 l -1 s 6 l -1 s -6 l -1

> torusUpperSlopes( 13, 5 )
[ 1/5 ], 11, 15, 21
> torusLowerSlopes( 13, 5 )
[ 1/3 ], 3, 3, 5, 5, 7, 7, 7, 9, 9

> print torusBraidWord( 13, 5 )
m 1 l -3 m 1 l -2 m 1 l -3 m 1 l -3
```

We are currently working on using the braid word method to understand tunnels of other interesting knots, such as the $(-2,3,7)$-pretzel knot.

How many (equivalence classes of) tunnels can a tunnel number 1 knot have? How many that are not $(1,1)$-tunnels?

Terminology: $(1,1)$-tunnels are called *semisimple,* non-$(1,1)$ tunnels are called *regular.*

Examples:

1. Knots with a unique regular tunnel

— This is the generic case for non-$(1,1)$ knots: By a theorem of M. Scharlemann and M. Tomova, if K has a tunnel of Heegaard distance at least 6, then it is the unique tunnel of K

2. $(1,1)$-knots with 2 tunnels— the upper and lower semisimple tunnels (these may be equivalent by a symmetry)

— Actually I don't know a fully proven example, but we expect that this is the generic case for $(1,1)$-knots

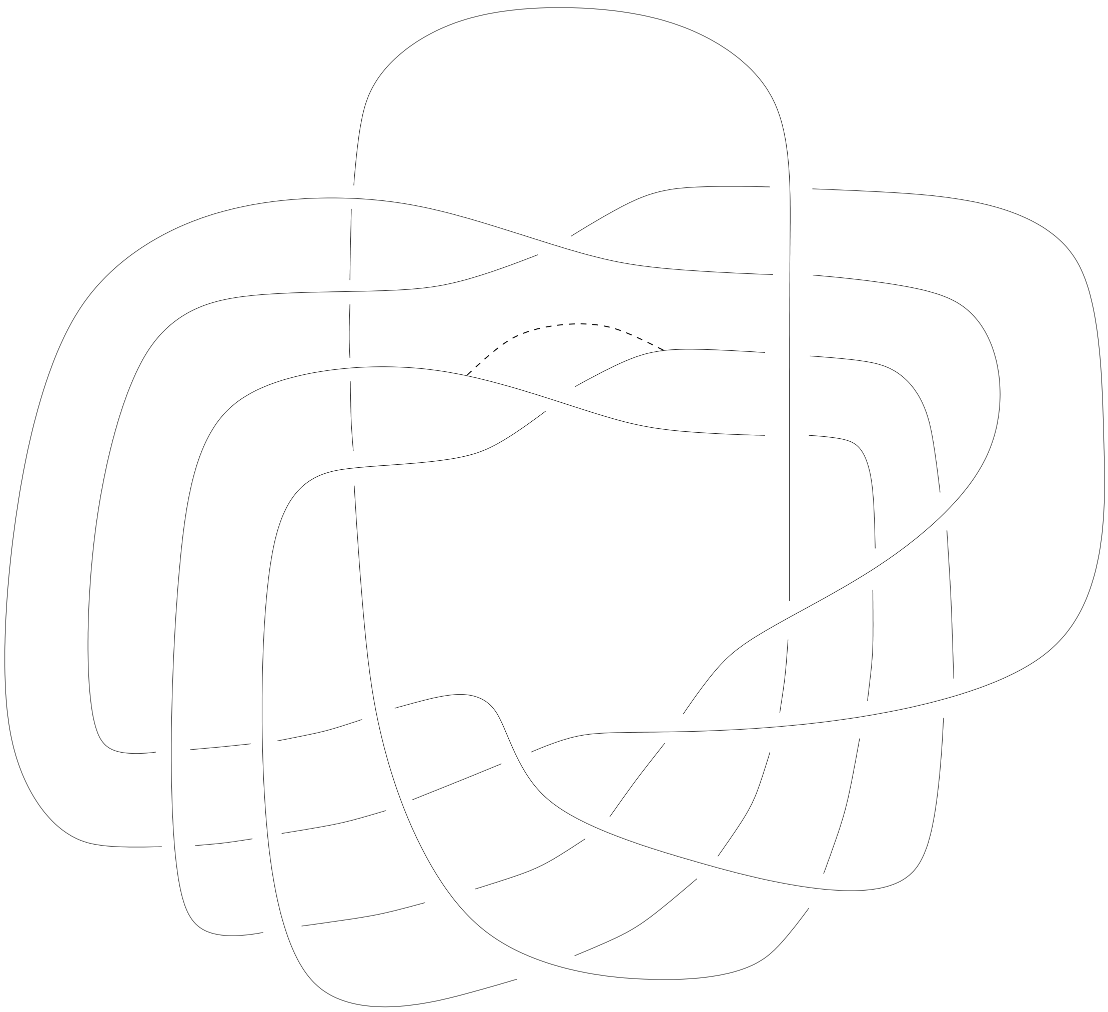
3. $(1,1)$-knots with two $(1,1)$-positions, and hence as many as 4 semisimple tunnels

— Examples include 2-bridge knots and the $(-2,3,7)$-pretzel knot

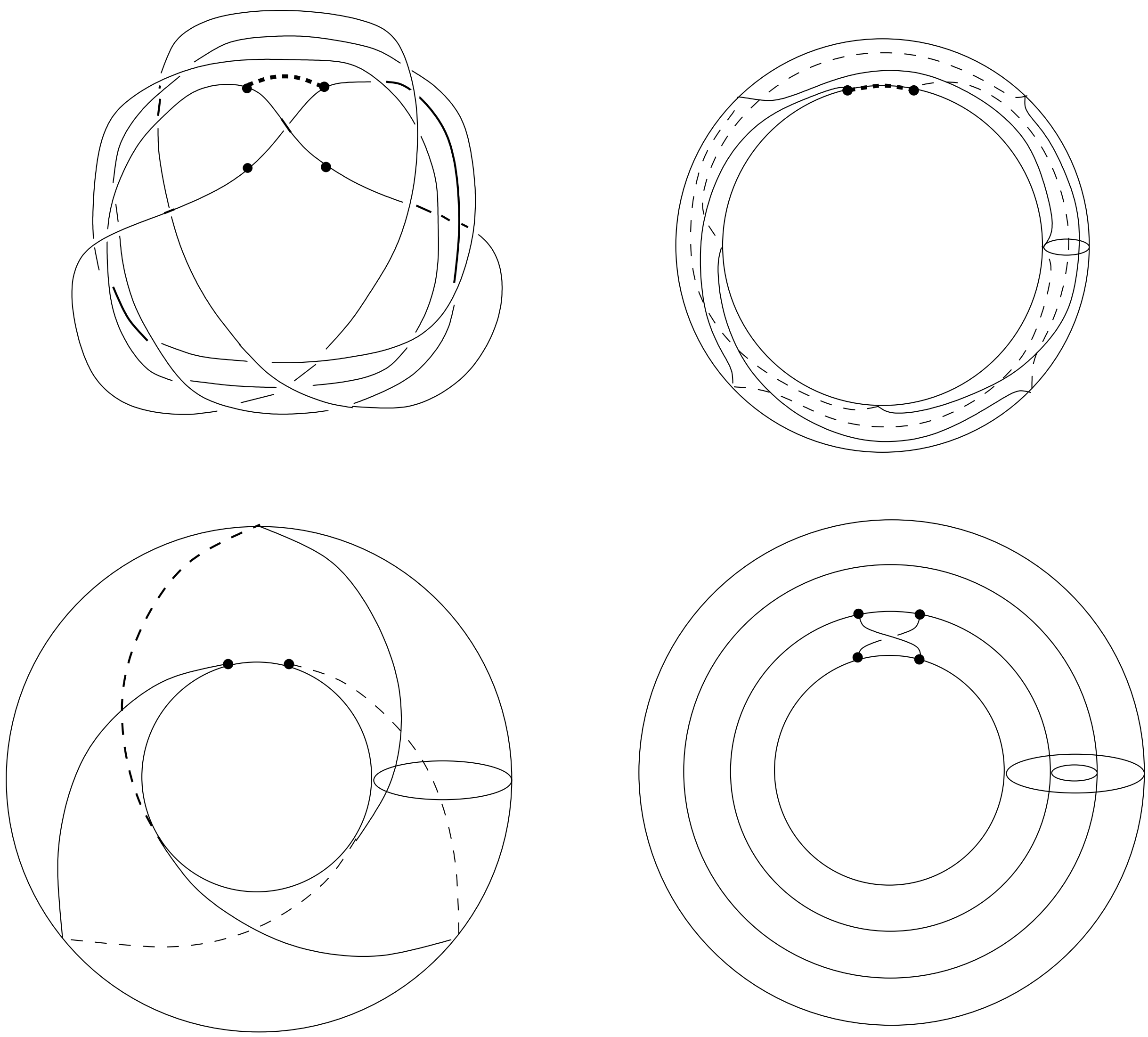
4. $(1,1)$-knots which have two semisimple tunnels and one regular tunnel

— Torus knots have (at most) three tunnels, 2 semisimples from the $(1,1)$-position and a third "middle" tunnel which is regular (except for certain cases where it is equivalent to one of the semisimple tunnels)

H. Goda and C. Hayashi recently found another example of a knot having both 2 semisimple tunnels and one regular tunnel. It is the Morimoto-Sakuma-Yokota $(5, 7, 2)$-knot, shown here with its regular tunnel:



As explained in the preprint of Goda and Hayashi, this knot can be contructed by taking a $(3, -4)$ torus knot and a $(2, -3)$ torus knot on cocentric tori and connecting them with half-twist. The regular tunnel goes horizontally between the half-twisted strands:



We understand the cabling sequence of the Goda-Hayashi example, and know exactly which pairs of torus knots can go on the two levels to obtain similar examples.

There is a way to iterate the construction to get examples with more than two levels, still in a $(1,1)$-position, having 2 semisimple tunnels and one regular tunnel.

We are currently studying these constructions, which we call collectively *Goda-Hayashi constructions.*

Here is a conjectural picture of knots with more than one tunnel— it is a very aggressive conjecture, since it is based mainly on the lack of counterexamples:

Conjecture: A tunnel number 1 knot K has at most 4 tunnels, and satisfies one of the following, allowing some of the tunnels to be equivalent due to symmetries:

1. K has a unique regular tunnel.

2. K has one $(1,1)$-position and 2 semisimple tunnels.

3. K has two $(1,1)$-positions and 4 semisimple tunnels.

4. K is a torus knot or a knot obtained by a Goda-Hayashi construction, and has 2 semisimple tunnels and 1 regular tunnel.