All tunnels of all tunnel number 1 knots

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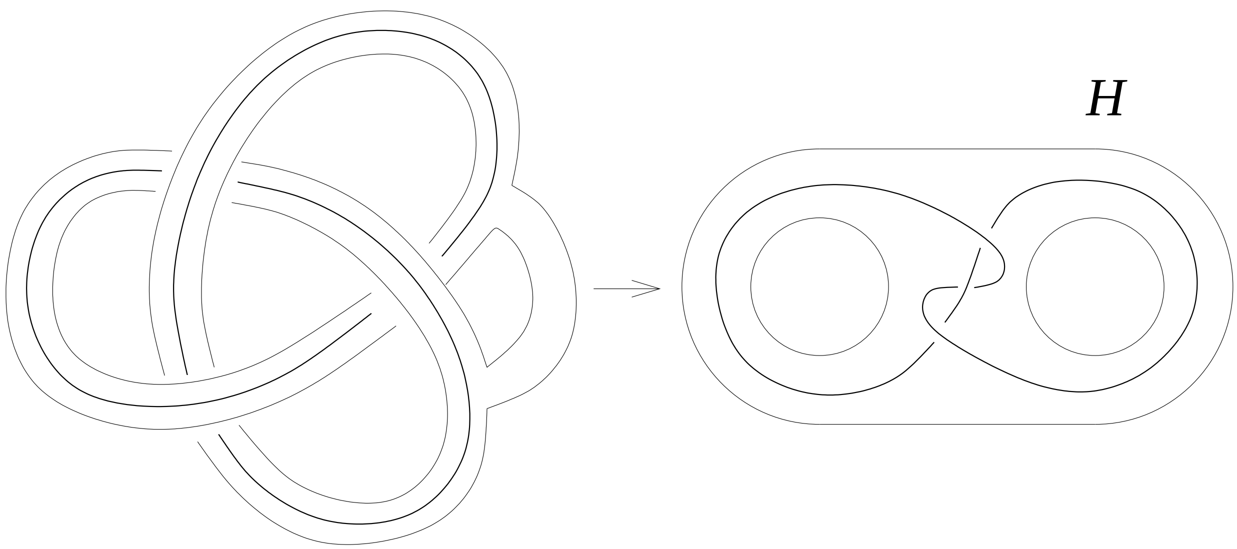
Geometric Topology Conference

Beijing University
June 22, 2007

(joint work with Sangbum Cho, in “The tree of knot tunnels”, ArXiv math.GT/0611921, and “The depth of a knot tunnel”, in preparation)

The classic picture:



A tunnel number 1 knot $K \subset S^3$ is a knot for which you can take a regular neighborhood of the knot and add a 1-handle in some way to get an unknotted handlebody (i. e. a handlebody which can be moved by isotopy to the standard handlebody H in S^3.)

The added 1-handle is called a *tunnel* of K.

Tunnels are *equivalent* when there is an orientation-preserving homeomorphism of S^3 taking knot to knot and tunnel to tunnel.

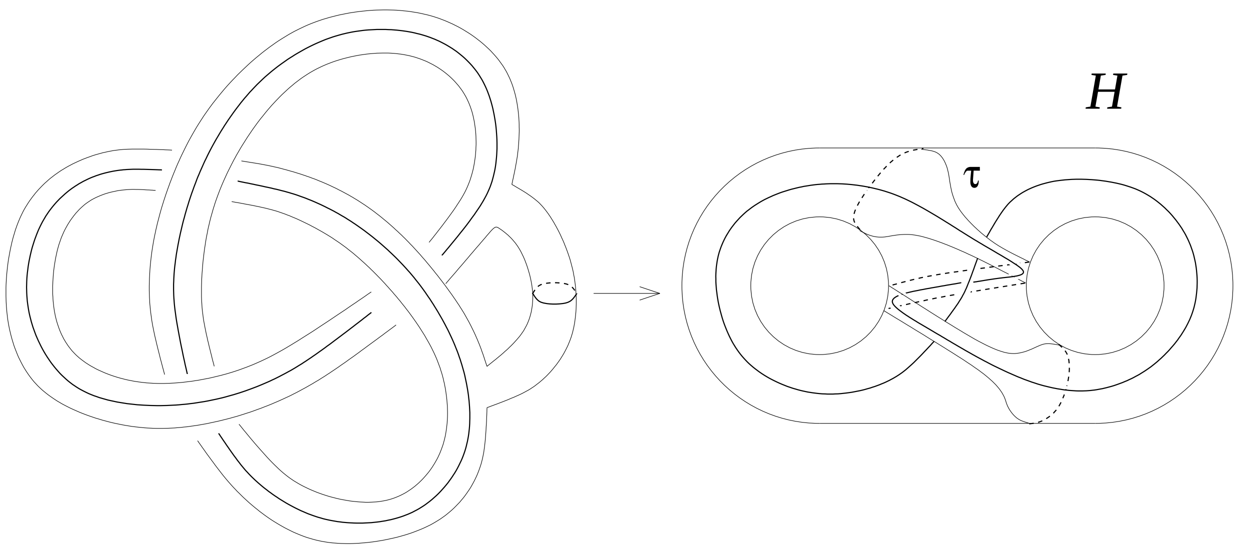
(There is also a concept of isotopy of tunnels. But all of our work uses only equivalence up to homeomorphism.)

If X is the knot space $\overline{S^3 - \mathrm{Nbd}(K)}$, then $H \cap X$ and $\overline{S^3 - H}$ form a genus-2 Heegaard splitting of the manifold-with-boundary X.

That is, X is decomposed into a compression body $H \cap X$ and a genus-2 handlebody $\overline{S^3 - H}$.

The equivalence classes of tunnels correspond to the homeomorphism classes of genus-2 Heegaard splittings of knot spaces. So the study of tunnel number 1 knots is the same as the study of genus-2 Heegaard splittings of knot spaces up to homeomorphism. But we will not use this viewpoint explicitly.

A natural idea is to examine a cocore 2-disk of the tunnel. An isotopy taking the knot and tunnel to H carries the cocore 2-disk to some disk τ in H.



Different isotopies moving the knot and tunnel to H may produce different disks in H. So each knot tunnel will produce some collection of *nonseparating* disks in H.

And each nonseparating disk τ in H is the cocore disk of a tunnel of some knot, in fact of the knot K_τ which is the core circle of the solid torus obtained by cutting H along τ.

To develop this idea, we must deal with two problems:

1. Understand the nonseparating disks in H.

2. Understand how changing the choice of isotopy to the standard H can change the disk we get in H.

Problem 1 was answered a long time ago. And Problem 2 is now answered by recent work of M. Scharlemann, E. Akbas, and S. Cho. We can combine this information to develop a new theory of tunnel number 1 knots.

Remark: For a tunnel of a tunnel number 1 *link*, the cocore disk of the tunnel is a separating disk. Our entire theory adapts easily to allow links instead of just knots. For simplicity, we will just talk about knots.

First, let's understand the nonseparating disks in the genus-2 handlebody H.

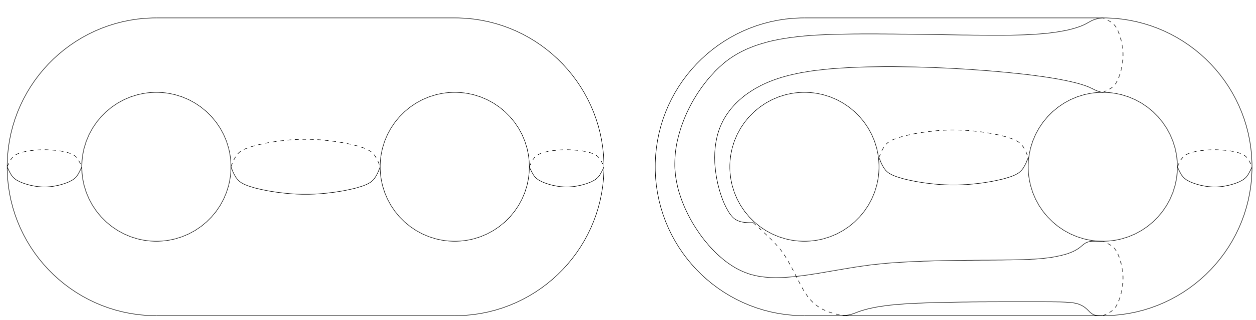
(From now on, "disk" will mean "nonseparating disk.")

Let $\mathcal{D}(H)$ be the complex of disks of H.

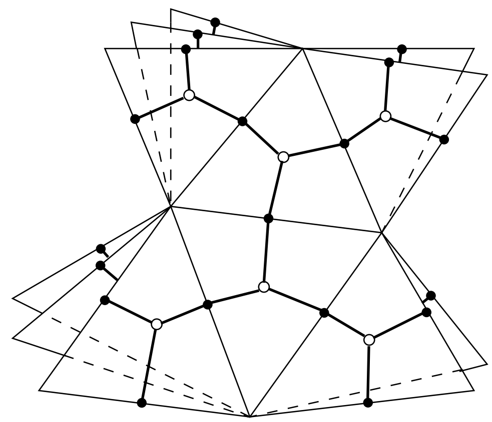
A vertex of $\mathcal{D}(H)$ is an isotopy class of properly-imbedded disks in H.

A collection of $k+1$ distinct vertices spans a k-simplex when one may select representative disks that are disjoint.

$\mathcal{D}(H)$ is 2-dimensional, because one can have at most 3 disjoint nonisotopic disks in H. Here are two 2-simplices that meet in an edge:



$\mathcal{D}(H)$ looks like this:

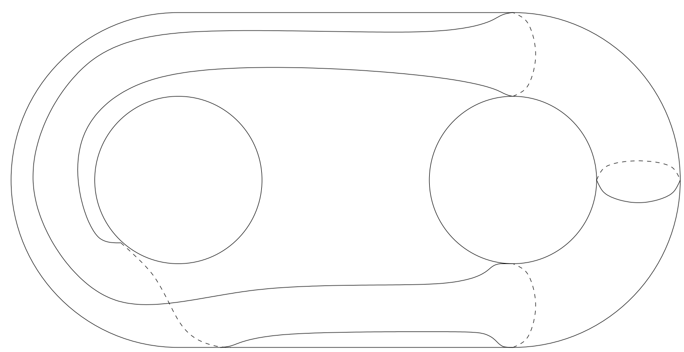


— $\mathcal{D}(H)$ has countably many 2-simplices attached along each edge

— $\mathcal{D}(H)$ is contractible (McC 1991, better proof Cho 2006). In fact, it deformation retracts to a bipartite tree T which has valence-3 vertices corresponding to triples of disks and countable-valence vertices corresponding to pairs of disks in H

Since H is the standard handlebody in S^3, $\mathcal{D}(H)$ has extra structure:

A disk $D \subset H$ is *primitive* if there exists a "dual" disk $D' \subset \overline{S^3 - H}$ such that ∂D and $\partial D'$ cross in one point. Here are two primitive disks in H:



One can prove that τ is primitive if and only if K_τ is the trivial knot in S^3. That is, *the primitive disks are exactly the disks that correspond to the trivial tunnel.*

The *Goeritz group* Γ is the group of orientation-preserving homeomorphisms of S^3 that preserve H, modulo isotopy through homeomorphisms preserving H.

Two isotopies moving a knot and tunnel to H differ by an element of Γ.

That is, the action of Γ is the indeterminacy of the disk obtained by moving the knot and tunnel to H.

Put differently,

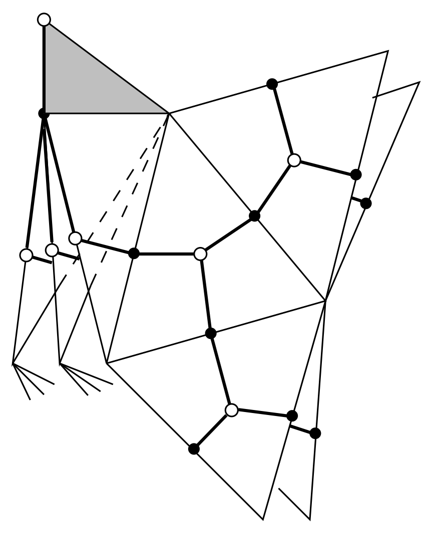
— Γ acts on $D(H)$, and
— the orbits of the vertices under this action correspond to the equivalence classes of all tunnels of all tunnel number 1 knots.

Therefore an equivalence class of tunnels corresponds to a single vertex of the quotient complex $D(H)/\Gamma$.

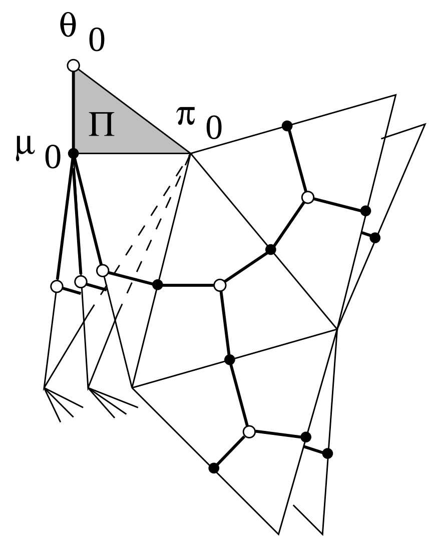
Theorem 1 *(M. Scharlemann, E. Akbas)* Γ *is finitely presented.*

This theorem was proven by delicate arguments using an action of Γ on a complex whose vertices are certain 2-spheres.

Cho reinterpreted their proof using the disk complex, and using his work we can completely understand the action of Γ on $\mathcal{D}(H)$, and describe the quotient $\mathcal{D}(H)/\Gamma$, which looks like this:



$\mathcal{D}(H)/\Gamma$ deformation retracts to the tree T/Γ.



Some other interesting features in $D(H)/\Gamma$ are:

1. π_0, the orbit of primitive disks, which represents the tunnel of the trivial knot.

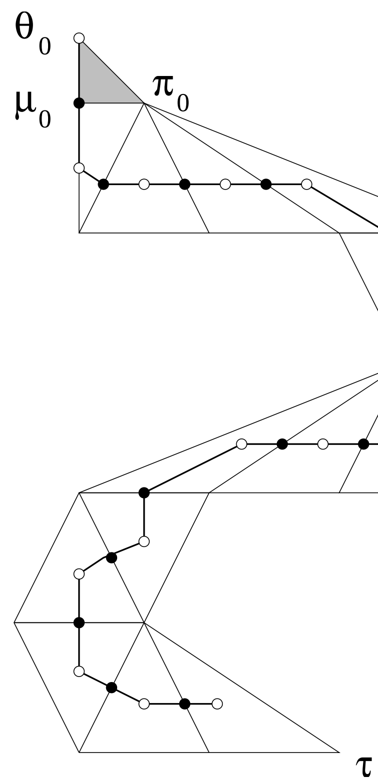
2. μ_0, the orbit of a primitive pair.

3. θ_0, the orbit of a primitive triple.

The last two are vertices of the tree T/Γ. The vertices that correspond to tunnels are those (like π_0) that are images of vertices of $D(H)$.

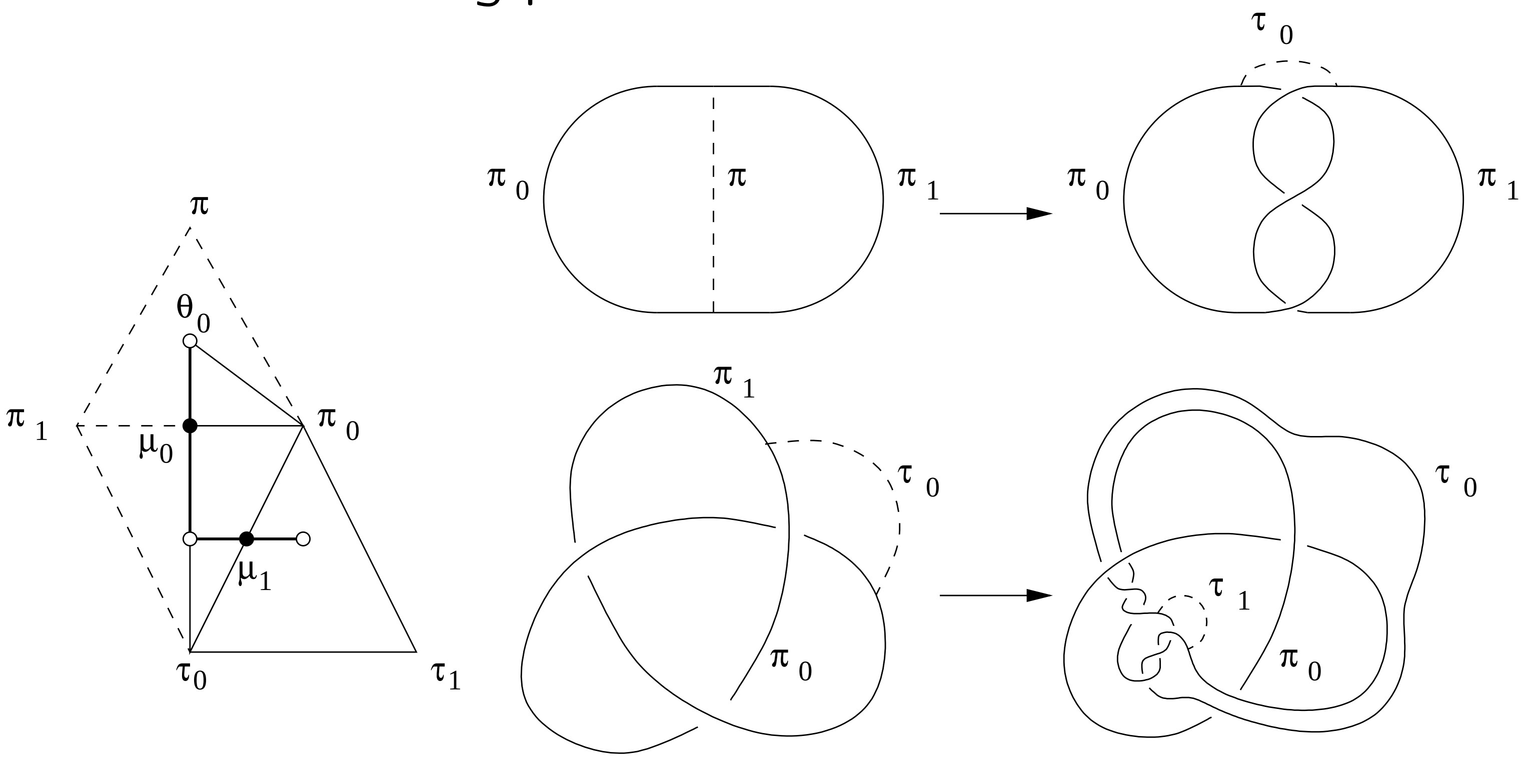
Fix a tunnel τ.

Since T/Γ is a tree, there is a unique path in T/Γ that starts at θ_0 and travels to the nearest barycenter of a simplex that contains τ. This is called the *principal path* of τ, shown here:



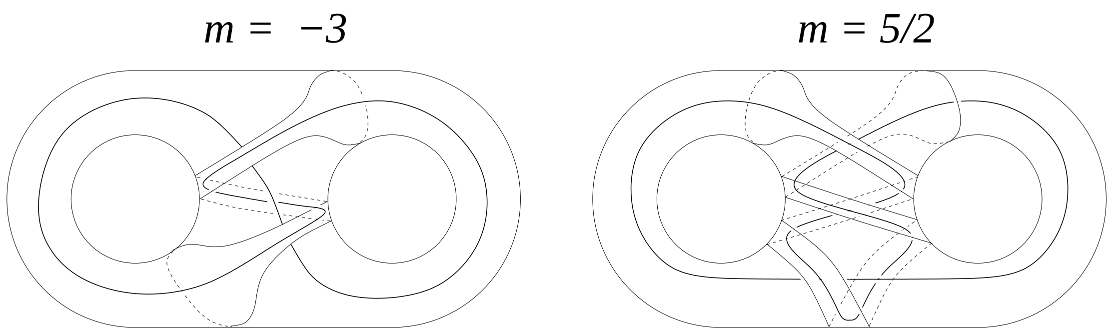
Traveling along the principal path of τ encodes a sequence of simple "cabling constructions" that produce new knots and tunnels, starting with the tunnel of the trivial knot and ending with τ.

The following picture indicates how this works:



Since T/Γ is a tree, every tunnel can be obtained by starting from the tunnel of the trivial knot and performing a *unique* sequence of cabling constructions.

A cabling operation is described by a rational "slope" parameter that tells which disk becomes the new tunnel disk (i. e. which of the countably many edges to take out of a black vertex).

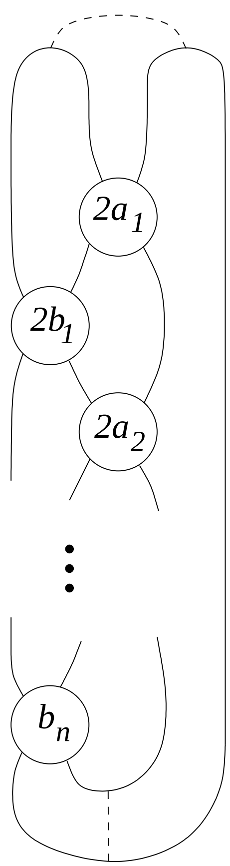


The slope of the *final* cabling operation is (up to details of definition) the tunnel invariant discovered by M. Scharlemann and A. Thompson.

The sequence of these slopes (plus a little bit more information telling which branch one takes at the white vertices), completely classifies the tunnel.

Let's look at the example of 2-bridge knots.

Roughly speaking, two-bridge knots are classified by a rational number (modulo $\mathbb{Z}$) whose reciprocal is given by the continued fraction with coefficients equal to the number of half-twists in the positions shown here:



$$\longleftrightarrow \quad 2a_1 + \cfrac{1}{2b_1 + \cfrac{1}{2a_2 + \cdots + \cfrac{1}{b_n}}}$$

$$= [\, 2a_1, 2b_1, \ldots, 2a_n, b_n \,]$$

The tunnels shown here are called the "upper" or "lower" tunnels of the 2-bridge knot.

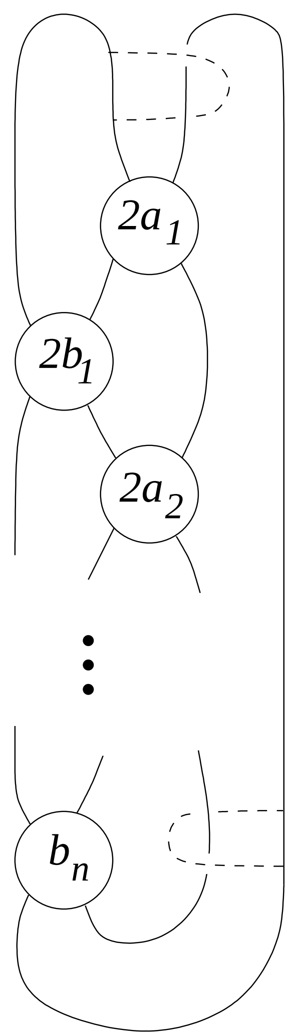
The upper and lower tunnels of 2-bridge knots are the tunnels that are obtained from the trivial knot by a *single* cabling operation. For technical reasons, the first slope parameter is only well-defined in $\mathbb{Q}/\mathbb{Z}$, and not surprisingly it is essentially the standard invariant that classifies the 2-bridge knot.

Our theory gives easy proofs of the following theorems about upper and lower tunnels:

Theorem 2 (D. Futer) *Let α be a tunnel arc for a nontrivial knot $K \subset S^3$. Then α is fixed pointwise by a strong inversion of K if and only if K is a two-bridge knot and α is its upper or lower tunnel.*

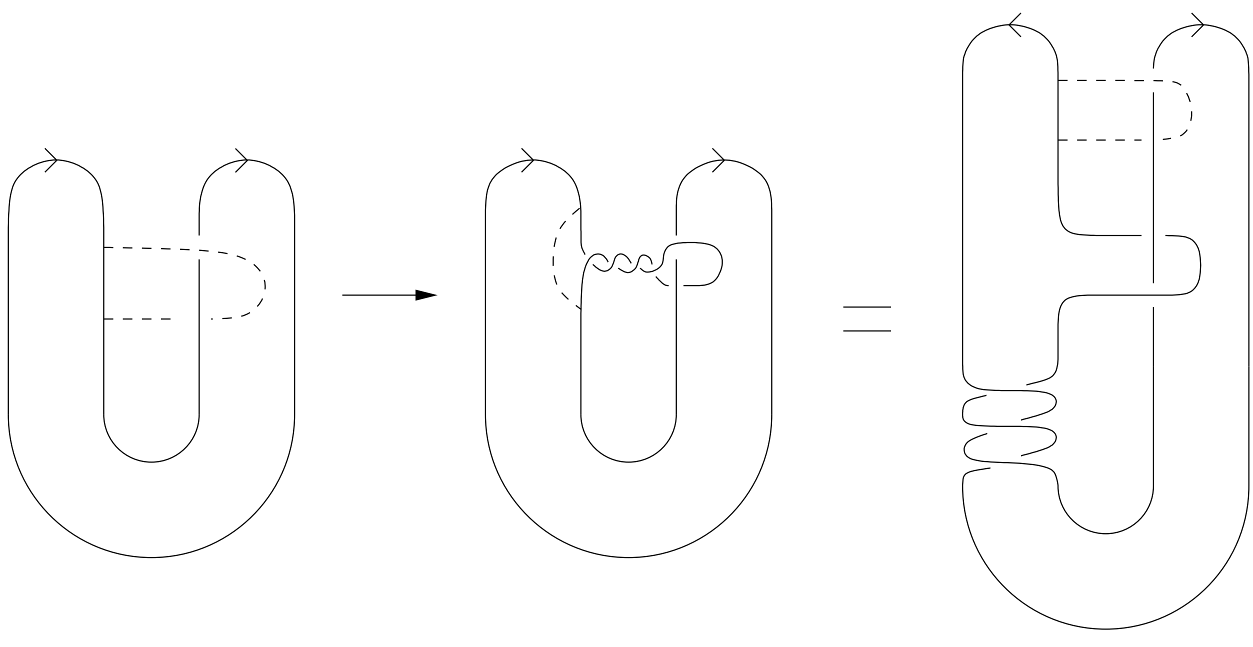
Theorem 3 (C. Adams-A. Reid, M. Kuhn) *The only tunnels of a 2-bridge link are its upper and lower tunnels.*

The other tunnels of 2-bridge knots were classified by T. Kobayashi, K. Morimoto, and M. Sakuma. Besides the upper and lower tunnels, there are (at most) two other tunnels, shown here:



For these other tunnels, the number of cablings equals the number of full twists of the middle two strands, that is, $a_1 + a_2 + \cdots + a_n$.

Each of these cablings adds one full twist to the middle two strands, but an arbitrary number of half-twists to the left two strands:



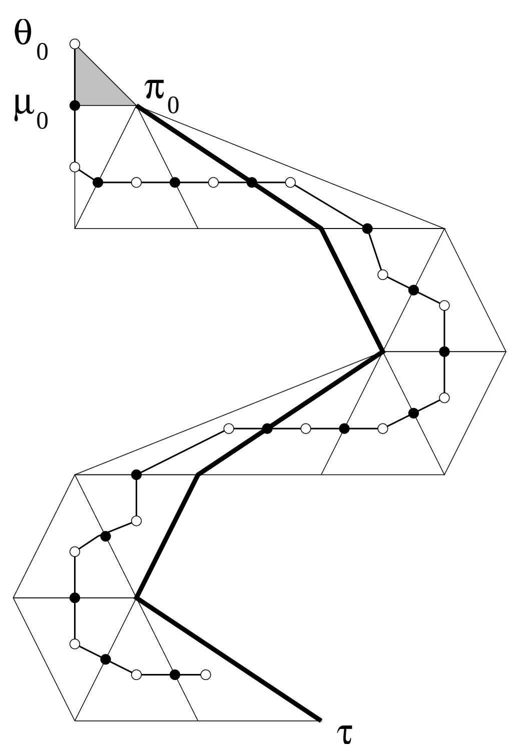
We have worked out an algorithm that starts with the classifying invariant of the 2-bridge knot, and obtains the slope parameters of the cablings in the cabling sequence of these other tunnels.

It is a bit complicated, but can be implemented computationally. Some sample output from the program:

```
TwoBridge> slopes (33/19)
[ 1/3 ], 3, 5/3

TwoBridge> slopes (64793/31710)
[ 2/3 ], -3/2, 3, 3, 3, 3, 3, 7/3, 3, 3, 3, 3, 49/24

TwoBridge> slopes (3860981/2689048)
[ 13/27 ], 3, 3, 3, 5/3, 3, 7/3, 15/8, -5/3, -1, -3

TwoBridge> slopes (5272967/2616517)
[ 5/9 ], 11/5, 21/10, -23/11, -131/66
```

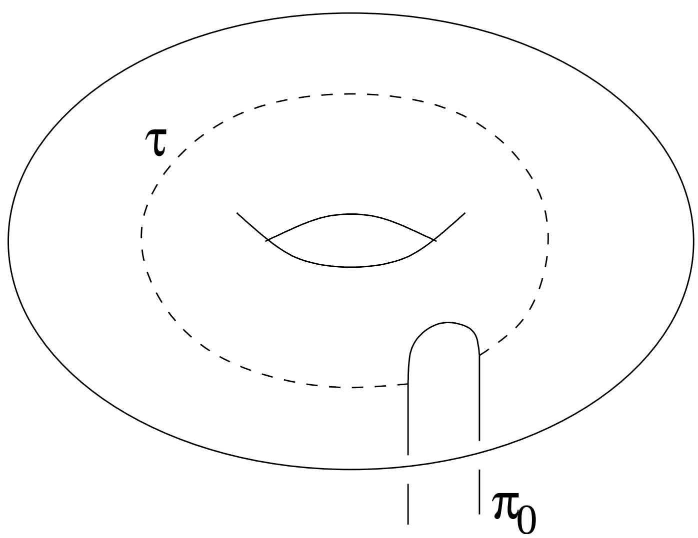
Some of the applications of our theory use a tunnel invariant called the *depth* of the tunnel. The depth of τ is the distance in the 1-skeleton of $D(H)/\Gamma$ from the (orbit of the) primitive disk π_0 to τ.

The tunnel that we saw earlier has depth 5:



The depth-1 tunnels are exactly the type usually called $(1,1)$-tunnels.

Their associated knots can be put into 1-bridge position with respect to a torus $\times\, I$ (genus-1 1-bridge position). A $(1,1)$-tunnel for a $(1,1)$-knot looks like this with respect to some $(1,1)$-position:

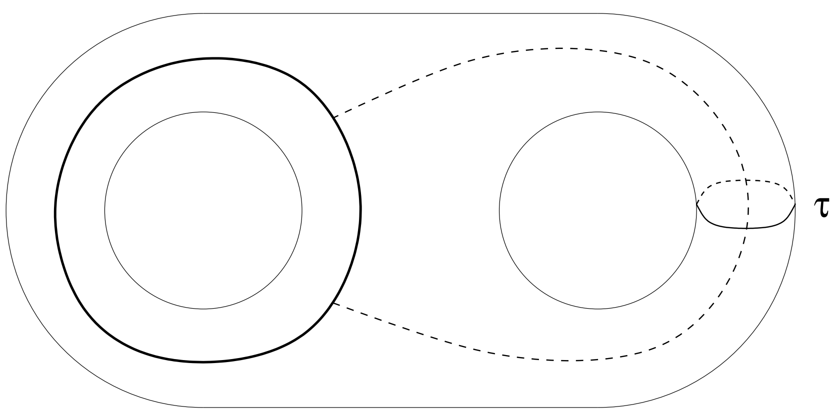


τ together with one of the arcs of the knot is an unknotted circle in S^3, so τ is disjoint from a primitive disk π_0, i. e. τ has depth 1. Conversely, it can be shown that every depth-1 tunnel is a $(1,1)$-tunnel.

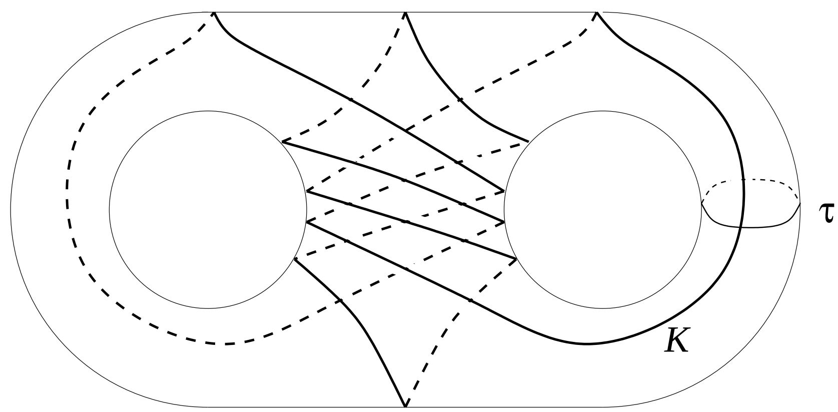
We saw that moving through T/Γ corresponds to constructing new tunnels by cabling constructions.

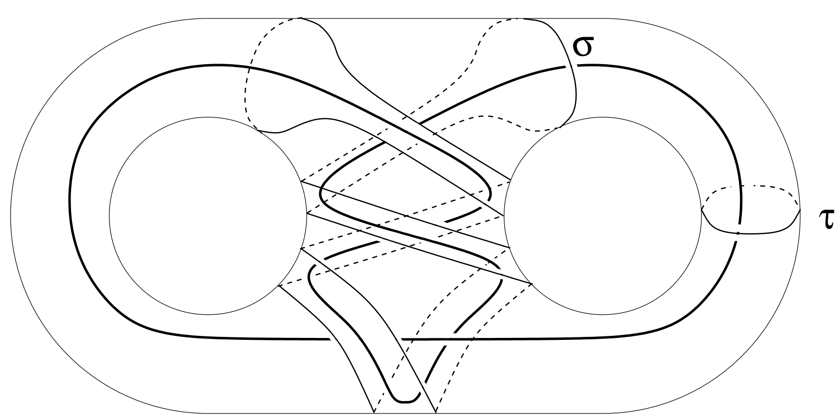
Moving through the 1-skeleton of $\mathcal{D}(H)/\Gamma$ also corresponds to a geometric construction of tunnels. It appears first in a paper of H. Goda, M. Scharlemann, and A. Thompson, and we call it a *GST-move.*

Start with a knot and a tunnel τ.

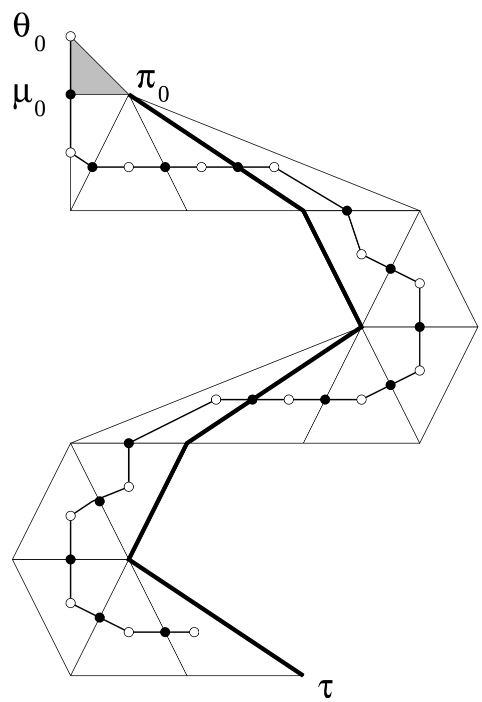


Choose any loop K in ∂H that crosses τ in exactly one point. It turns out that this must be a tunnel number 1 knot with a tunnel disk σ disjoint from τ.



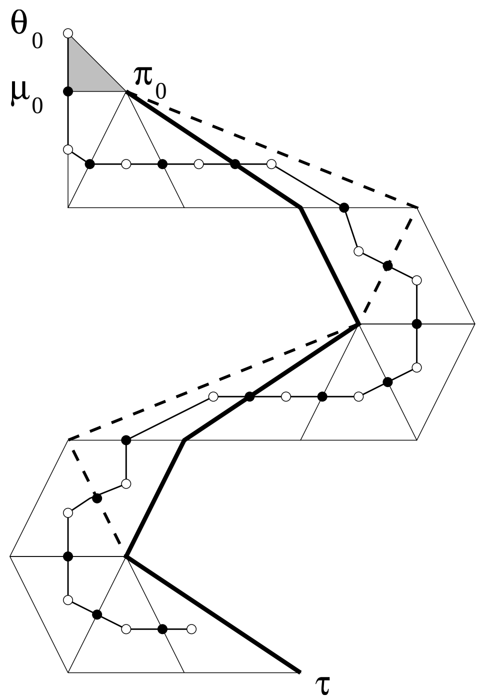


In $\mathcal{D}(H)/\Gamma$, this GST-move corresponds to moving along the 1-simplex from τ to σ.



Thus our depth 5 tunnel can be obtained from the trivial tunnel by 5 GST-moves, and this is the minimal number possible.

GST-moves can have a much more drastic effect than cabling constructions— this example requires 15 cabling constructions. Also, any $(1,1)$-tunnel is produced from the trivial tunnel by a single GST-move.

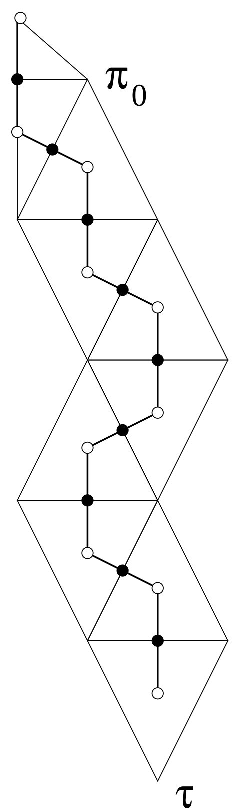


The minimal GST-sequence producing a given tunnel is usually not unique. In this example, there are two places where another route is possible, leading to four possible minimal GST-sequences producing τ.

It is a combinatorial exercise to work out an algorithm for the number of minimal paths from π_0 to τ in the 1-skeleton of $\mathcal{D}(H)/\Gamma$, and hence the number of minimal GST-move constructions of a tunnel.

For a sparse infinite set of tunnels, there is a unique minimal GST-construction sequence.

In contrast, for example, this depth-5 tunnel has 8 minimal GST-constructions:



If one continues in this same pattern, the first depth-n tunnel in this sequence has a_n minimal GST-constructions, where

$$(a_0, a_1, a_2, a_3, a_4, \ldots) = (1, 1, 2, 3, 5, 8, 13, \ldots)$$

Using the "tunnel leveling" theorem of H. Goda-M. Scharlemann-A. Thompson, as applied to GST-moves in their original paper, we can find the minimum bridge number of K_τ as a function of depth(τ).

Theorem 4 *For $d \geq 1$, the smallest bridge number of a knot having a tunnel of depth d is b_{2d}, where b_n is the sequence given by the recursion*

$$b_2 = 2, b_3 = 2$$
$$b_{2n} = b_{2n-1} + b_{2n-2}$$
$$b_{2n+1} = b_{2n} + b_{2n-2}$$

Corollary 1 *For any sequence of tunnels, the asymptotic growth rate of the bridge number of K_τ as a function of* depth(τ) *is at least proportional to $(1+\sqrt{2})^d$, and this rate is best possible, in general.*

Another measure of complexity for a tunnel has been studied by J. Johnson, A. Thompson, Y. Minsky-Y. Moriah-S. Schleimer, and others:

The *(Hempel) distance* $\mathrm{dist}(\tau)$ is the distance in the *curve complex* $\mathcal{C}(\partial H)$ from $\partial\tau$ to a loop that bounds a disk in $\overline{S^3 - H}$.

Distance is related to depth by $\mathrm{dist}(\tau) - 1 \leq \mathrm{depth}(\tau)$ (so $(1+\sqrt{2})^d$ is also a lower bound for the growth rate of bridge number as a function of distance).

But depth is a finer invariant than distance:

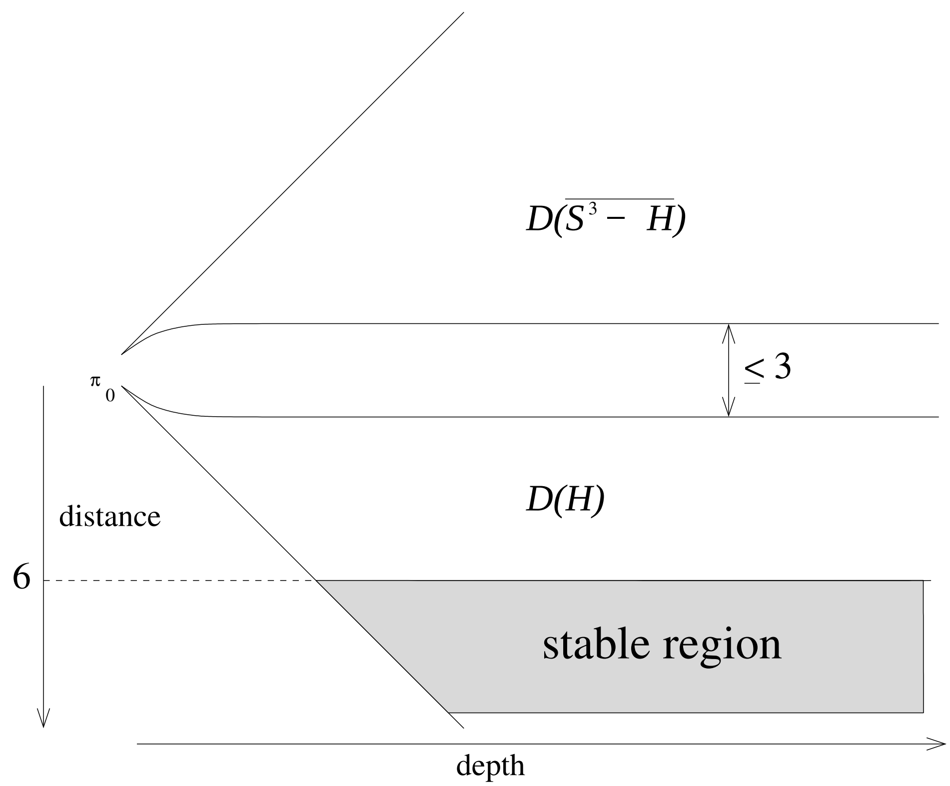
The "short" tunnels of torus knots all have distance 3, but their depths can be arbitrarily large (the depth of the short tunnel of the (p,q)-torus knot is approximately the number of terms in the continued fraction expansion of p/q).

It would be very interesting to understand better the relation between depth and distance. Recent work of S. Schleimer

— When does a cabling operation that increases depth also increase distance?

— In particular, is there a cabling construction of *non-integral* slope that raises depth but does not raise distance?
(For the large-depth small-distance examples of torus knot tunnels, all the cabling constructions have integral slope.)

The disk complex imbeds in the curve complex $\mathcal{C}(\partial H)$, just by taking each D to ∂D. Here is a schematic picture:



The "stable region" is the region of tunnels of distance at least 6. J. Johnson, using results of M. Scharlemann and M. Tomova, proved that

Theorem 5 *If K has a tunnel of distance at least 6, then this tunnel is the unique tunnel of K up to isotopy.*

It appears that much of the complicated behavior of tunnel number 1 knots appears at depth 1, i. e. the $(1,1)$-tunnels.

For example, as far as I know there is no known example of a knot that has more than one equivalence class of tunnels of depth greater than 1.

— Most torus knots have three equivalence classes, two of depth 1 and the other of larger depth.
— For the other known examples of knots with multiple equivalence classes of tunnels (2-bridge knots, some pretzel knots, etc.), all tunnels are depth 1.

Conjecture: No knot has more than one tunnel of depth larger than 1.

A few words about knots of tunnel number ≥ 2:

The analogous theory for knots of tunnel number larger than 1 would involve the disk complexes of higher genus handlebodies. For genus g, the disk complex is $(3g-4)$-dimensional. Although these are contractible, their structures seem much more difficult to understand than for the genus-2 case.

It also appears to be much more difficult to understand the subcomplex of primitive disks, or even to be sure what to use as the concept of primitivity.

In fact, for genus ≥ 3, it has not even been proven that the Goeritz group is finitely generated. However, very recent work D. Bachman and S. Schleimer appears to give a proof that the complex of reducing spheres is connected, which should imply the finite generation.