

Nielsen equivalence, group actions, and $\text{PSL}(2, q)$

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Let G be a finitely generated group.

Denote by $\mathcal{G}_k(G)$ the set of *generating k -vectors*,

$$\mathcal{G}_k(G) = \{(g_1, \dots, g_k) \mid \langle g_1, \dots, g_k \rangle = G\} .$$

Several relations can be defined on $\mathcal{G}_k(G)$:

- (i) *Product replacements*: $(g_1, \dots, g_i, \dots, g_j, \dots, g_k) \sim (g_1, \dots, g_i g_j, \dots, g_j, \dots, g_k)$ (or instead of $g_i g_j$, one of $g_i g_j^{-1}$, $g_j g_i$, or $g_j^{-1} g_i$).
- (ii) permute the entries and/or replace some of them by their inverses
- (iii) $(g_1, \dots, g_k) \sim (\alpha(g_1), \dots, \alpha(g_k))$, where $\alpha \in \text{Aut}(G)$.

These generate equivalence relations on $\mathcal{G}_k(G)$:

- (i) \cup (ii) generate *Nielsen equivalence* (\sim_N).
- (i) \cup (ii) \cup (iii) generate *T -equivalence* (\sim_T).

The equivalence classes for Nielsen equivalence are called *Nielsen classes*, and those for T -equivalence are called *T -systems*.

I don't know the exact history of these, but among the early researchers who developed them are *B. H.* and *Hanna Neumann*.

One reason these equivalence relations arise naturally is that they classify certain extensions, as follows. Let F_k be the free group on a set of k elements $\{x_1, \dots, x_k\}$. Now, there is a bijective correspondence $\mathcal{G}_k(G) \longleftrightarrow \text{Epi}(F_k, G)$, from the k -element generating vectors of G to the surjective homomorphisms from F_k to G , defined by sending (g_1, \dots, g_k) to the homomorphism π that sends each x_i to g_i . Each element of $\text{Epi}(F_k, G)$ determines an extension $1 \rightarrow \ker(\pi) \rightarrow F_k \xrightarrow{\pi} G \rightarrow 1$, and there are equivalence relations on these extensions defined by commutative diagrams:

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & \ker(\pi) & \longrightarrow & F_k & \xrightarrow{\pi} & G & \longrightarrow & 1 \\
 \downarrow & & \downarrow \phi|_{\ker(\pi)} & & \downarrow \phi & & \downarrow \alpha & & \downarrow \\
 1 & \longrightarrow & \phi(\ker(\pi)) & \longrightarrow & F_k & \xrightarrow{\pi'} & G & \longrightarrow & 1
 \end{array}$$

where $\pi \sim_T \pi'$ if $\pi' = \alpha \circ \pi \circ \phi^{-1}$ for some $\alpha \in \text{Aut}(G)$ and some $\phi \in \text{Aut}(F_k)$, and where $\pi \sim_N \pi'$ if α can be taken to be id_G . Using Nielsen's result that the moves (i) and (ii) applied to (x_1, \dots, x_k) generate $\text{Aut}(F_k)$, it is straightforward to check that these equivalence relations on $\text{Epi}(F_k, G)$ correspond to T -equivalence and Nielsen equivalence in $\mathcal{G}_k(G)$.

Notice that this shows that the Nielsen classes in $\mathcal{G}_k(G)$ are exactly the orbits of the right action of $\text{Aut}(F_k)$ on $\text{Epi}(F_k, G)$.

Here are several examples.

1. $G = C_n = \langle t \mid t^n = 1 \rangle$, $n > 1$.

$\mathcal{G}_1(C_n) = \{(t^m) \mid \gcd(m, n) = 1\}$. This has $\varphi(n)$ elements, where the Euler function $\varphi(n)$ equals the number of m with $1 \leq m < n$ and $\gcd(m, n) = 1$. The only nontrivial Nielsen equivalence is that $(t^m) \sim_N (t^{n-m})$, so there are $\varphi(n)/2$ Nielsen classes. Whenever $\gcd(m, n) = 1$, there is an automorphism of C_n defined by sending t to t^m , so $(t) \sim_T (t^m)$ and $\mathcal{G}_1(C_n)$ has only one T -system.

2. $G = C_5$.

We have already seen that $(t) \not\sim_N (t^2)$. But we have

$$(t, 1) \sim_N (t, t^2) \sim_N (t \cdot (t^2)^2, t^2) = (1, t^2) \sim_N (t^2, 1).$$

In fact, one can check very easily that $\mathcal{G}_2(C_5)$ has only one Nielsen class. It is conjectured that this happens very generally:

Conjecture: For G finite and k larger than the minimum number of elements in a generating set of G , $\mathcal{G}_k(G)$ has only one Nielsen class.

This is known to be false for G infinite, but has been proven true for all (finite or infinite) solvable G (M. Dunwoody), and for $\mathrm{PSL}_2(p)$ and various other simple groups (R. Gilman, M. Evans).

3. $G = A_5$, $k = 2$.

This case was originally studied by B. H. and Hanna Neumann. We will explain that there are 3 Nielsen classes, represented by the pairs $((1, 2, 3, 4, 5), (1, 2, 4))$, $((1, 2, 3, 5, 4), (1, 2, 5))$, and $((1, 2, 3, 4, 5), (1, 2, 3, 5, 4))$. The first two are T -equivalent by applying the automorphism of A_5 that conjugates by $(4, 5)$, and there are two T -systems.

By playing around with permutations, it is not very hard to show that every generating pair is Nielsen equivalent to one of these three. But

proving that no two of these three pairs are *not* Nielsen equivalent requires an idea, an important one called the *Higman invariant*. It is the observation that for a generating pair (A, B) of a 2-generator group G , product replacements change the commutator $[A, B]$ only by conjugacy, transposing A and B or replacing one of them by its inverse changes $[A, B]$ to a conjugate of $[B, A]$, and applying $\alpha \in \text{Aut}(G)$ changes $[A, B]$ to $[\alpha(A), \alpha(B)] = \alpha([A, B])$. So the pair of conjugacy classes of $[A, B]$ and $[B, A]$ (which are possibly the same conjugacy class) is an invariant of the Nielsen class of (A, B) , and the orbit of these conjugacy classes under the action of $\text{Aut}(G)$ is an invariant of the T -system of (A, B) .

One can easily compute the Higman invariants of these three generating pairs of A_5 to see that they are not Nielsen equivalent (and obtain the result on T -systems by similar reasoning), but we prefer to see it in a way more related to some of the work we will discuss later. Regard A_5 as $\text{PSL}(2, 4)$ (recall that $\text{PSL}(2, q)$ is the group of 2×2 matrices with entries in the field \mathbb{F}_q with q elements and determinant 1, modulo the subgroup $\pm I$). Write \mathbb{F}_4 as $\{0, 1, \mu_1, \mu_2\}$. It turns out that the Higman invariants of these three pairs have traces μ_1, μ_2 , and 1, and since the trace is invariant under conjugation, this shows that the pairs are not Nielsen equivalent. It is also known that $\text{Aut}(\text{PSL}(2, q))$ is generated by conjugation by elements of $\text{GL}(2, q)$ and by applying field automorphisms of \mathbb{F}_q to the matrix entries. These change the trace of $[A, B]$ only by a field automorphism. The only nontrivial field automorphism of $\text{PSL}(2, 4)$ is the one that interchanges μ_1 and μ_2 , showing that the first two generating pairs cannot be T -equivalent to the third one.

The previous example is related to some results of R. Guralnick and I. Pak, published in PAMS in 2002. They used representation theory to show

- 1) For $k \geq 3$ there is no invariant word (such as $[A, B]$) for $k = 3$.
(Conjecturally, $[A, B]$ is the only such invariant word for $k = 2$.)
- 2) As primes $p \rightarrow \infty$, the number of T systems of $\mathrm{PSL}(2, p)$ goes to ∞ .

We will have more to say about the result 2. a bit later.

Nielsen equivalence has various applications in topology and algebra. We will mention a few here:

1. Algebra problem: Given a finite group G , generate random elements of G .

The best known algorithm for this seems to be the following one introduced by Leedham-Green and Soicher: Start with an element of $\mathcal{G}_k(G)$ (for k somewhat larger than the minimum number of elements of G), apply t random product replacements (say, for t some fixed number quite a bit larger than k), and take a random entry. This is the standard routine used in GAP and MAGMA.

It is not fully understood why this algorithm works so well in practice, but it is the object of a lot of interesting research in computational group theory. See the excellent survey by I. Pak, in *Groups and computation, III (Columbus, OH, 1999)*, 301–347, de Gruyter, Berlin, 2001; MR1829489 (2002d:20107).

2. Heegaard splittings

If a 3-manifold M has a Heegaard splitting $M = V \cup W$, where V and W are genus- g handlebodies, this splitting determines a Nielsen class in $\mathcal{G}_k(\pi_1(M))$. For the inclusion induces a homomorphism $F_k \cong \pi_1(V) \rightarrow \pi_1(M)$ whose Nielsen class is well-defined. In fact, since conjugating all elements of a generating vector by the same element of gives a Nielsen equivalent vector, isotopic Heegaard splittings give Nielsen equivalent elements of $\mathcal{G}_k(\pi_1(M))$. In a series of papers, M. Lustig and Y. Moriah have used Nielsen equivalence to obtain results about equivalent and inequivalent Heegaard splittings of various 3-manifolds. Using the Fox differential calculus, they developed an algebraic invariant (an equivalence class of matrices in a group somewhat like the Whitehead group) and used it to detect inequivalent Heegaard splittings.

3. Free G -actions on handlebodies.

In joint work with M. Wanderley (Free actions on handlebodies, J. Pure Appl. Algebra. 181 (2003), 85-104), we used Nielsen equivalence to classify free G -actions on handlebodies. By a free G -action, we mean an imbedding $\phi: G \rightarrow \text{Diff}_+(V)$ of a finite group into the group of orientation-preserving diffeomorphisms of an orientable 3-dimensional handlebody V . The quotient V/G must be a handlebody, and a simple Euler characteristic calculation shows that the genera of V and V/G are related by the formula $\chi(V) = 1 + |G|(\chi(V/G) - 1)$. Denote the genus of V/G by k . Regarding G as a group of covering transformation, the theory of covering spaces gives an extension

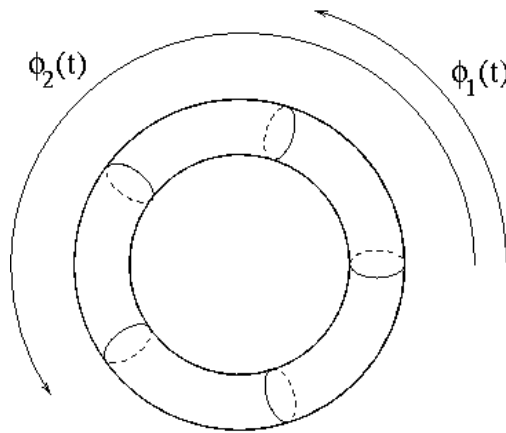
$$1 \rightarrow \pi_1(V) \rightarrow \pi(V/G) \rightarrow G \rightarrow 1 .$$

Fixing an isomorphism $\pi_1(V) \cong F_k$, we can regard this as a Nielsen equivalence class in $\mathcal{G}_k(G)$. It can be shown, just using covering space theory and the fact that $\text{Diff}(V) \rightarrow \text{Out}(\pi_1(V))$ is surjective,

that this sets up a bijection from equivalence classes (respectively, weak equivalence classes) of free G actions with quotient of genus k , and Nielsen classes (respectively, T -systems) in $\mathcal{G}_k(G)$.

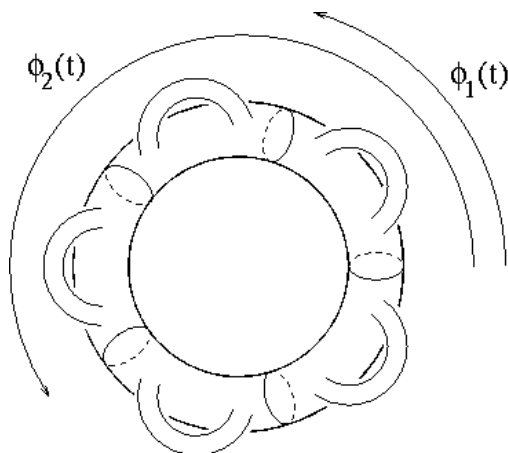
(Definition: ϕ_1 and ϕ_2 are *weakly equivalent* if there exist a diffeomorphism $h: V \rightarrow V$ and an automorphism $\alpha: G \rightarrow G$ such that $\phi_1(\alpha(g)) = h\phi_2(\alpha(g))h^{-1}$ for every $g \in G$, and are *equivalent* if they are weakly equivalent and α can be taken to be the identity on G . Equivalent actions are the same after change of coordinates on V , and weakly equivalent actions are equivalent after change of G by automorphism. Algebraically, equivalent actions are conjugate representations of G in $\text{Diff}(V)$, and weakly equivalent actions are representations with conjugate images in $\text{Diff}(V)$.)

Some examples should help make this theory more concrete. First, consider the following two C_5 -actions on V_1 :

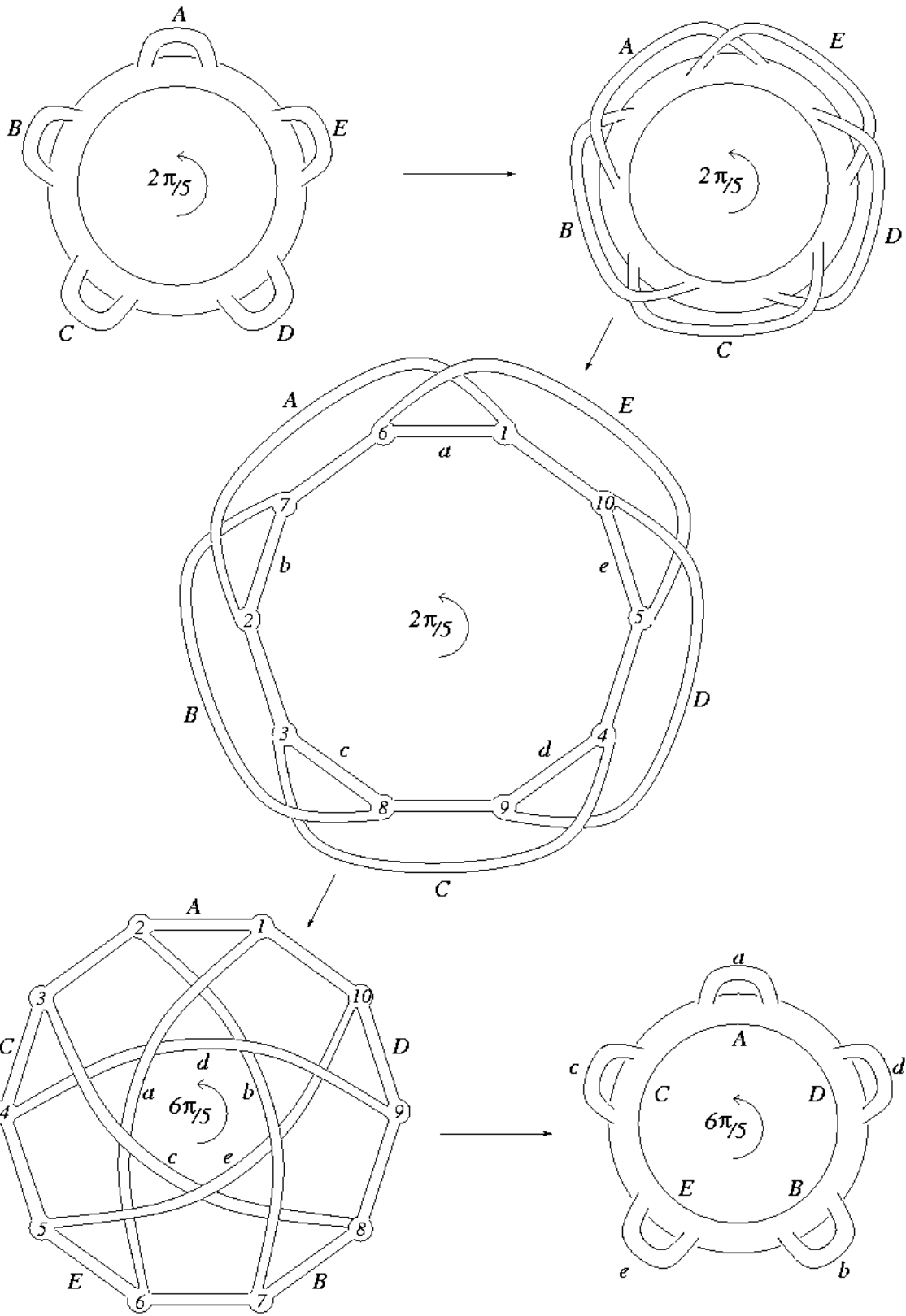


For ϕ_1 , the element $\phi_1(t)$ rotates through an angle of $2\pi/5$, while $\phi_2(t)$ rotates through an angle of $6\pi/5$. Under the correspondence of equivalence classes of free C_5 -actions with quotient of genus 1 and Nielsen classes in $\mathcal{G}_1(C_5)$, one has: $[\phi_1] \leftrightarrow [(t)]$ and $[\phi_2] \leftrightarrow [(t^2)]$. (You might be expecting $[\phi_2] \leftrightarrow [(t^3)]$, but the correspondence takes ϕ_i to (t^j) where $\phi_i(t^j)$ is the covering transformation on V_1 determined by $1 \in \mathbb{Z} \cong \pi_1(V_1/C_5)$, that is, j is the power of $\phi_i(t)$

which rotates the solid torus through $2\pi/5$.) Now, $(t) \not\sim_N (t^2)$, which tells us that ϕ_1 and ϕ_2 are not equivalent. However, $(t) \sim_T (t^2)$, since if $\alpha: t \mapsto t^2$ then $\phi_2 \circ \alpha = \phi_1$, so the actions are weakly equivalent. Now, consider similar actions on V_6 :



This time, recalling example 2 above, we have $[\phi_1] = [(t, 1)] \sim_N [(t^2, 1)] = [\phi_2]$, so ϕ_1 and ϕ_2 are equivalent! This may seem hard to believe, at first. The figure on the next page shows an explicit equivalence, by giving a sequence of C_5 -equivariant homeomorphisms of V_6 sending one action to the other. The first (top left) picture represents the action ϕ_1 , for which $\phi_1(t)$ sends A to B , B to C , and so on. Going from the first to the second is an equivariant slide of the left ends of the five 1-handles. The next arrow is just a redrawing of the second picture as a handle decomposition with ten 0-handles marked 1 through 10 and 15 1-handles, five marked A through E , five marked a through e , and five unmarked. The fourth picture is the same picture, but drawn with a different solid torus as the “long” one, and the fifth is the fourth one after sliding the bases of the handles. The same homeomorphism that moved A to B , corresponding to $\phi_1(t)$ in the first picture, now corresponds to $\phi_2(t)$.



We will close by describing some ongoing joint work with M. Wanderley. It concerns T -systems (with $k = 2$) of the groups $\mathrm{PSL}(2, q)$. Consider the following diagram:

$$\begin{array}{ccc}
 N\text{-classes} & \xrightarrow{[(A,B)] \mapsto \mathrm{tr}([A,B])} & \mathbb{F}_q \\
 \downarrow & & \downarrow \\
 T\text{-systems} & \longrightarrow & \mathrm{Aut}(\mathbb{F}_q)\text{-orbits in } \mathbb{F}_q
 \end{array}$$

The top horizontal arrow is the *trace invariant*. The bottom horizontal arrow is induced since $\mathrm{Aut}(\mathbb{F}_q)$ is known to be generated by conjugations by elements of $\mathrm{GL}(2, q)$, which do not change $\mathrm{tr}([A, B])$ at all, and applications of field automorphisms to the entries, which change $\mathrm{tr}([A, B])$ by the field automorphism. The bottom arrow we call the *weak trace invariant*.

There are many arguments showing that if (A, B) generates $\mathrm{SL}(2, q)$, then $\mathrm{tr}([A, B]) \neq 2$. This is basically the only field element that is not a trace invariant:

Proposition 1. *For $q \geq 13$, every element of \mathbb{F}_q other than 2 is the trace of $[A, B]$ for some generating pair of $\mathrm{PSL}(2, q)$.*

Proof. One calculates that

$$\mathrm{tr} \left(\left[\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}, \begin{bmatrix} y+1 & 1 \\ y & 1 \end{bmatrix} \right] \right) = 2 - (x - x^{-1})y .$$

When $q \geq 13$, this pair of matrices can be shown to generate $\mathrm{PSL}(2, q)$ whenever $\langle x \rangle = \mathbb{F}_q - \{0\}$ and $y \neq 0$. So every value except 2 occurs as the trace of some generating pair. \square

When $q \leq 11$, there are a few cases of other values of \mathbb{F}_q that do not occur as trace invariants (as we have already seen for $\mathrm{PSL}(2, 4) = \mathrm{PSL}(2, 5) = A_5$). These can be worked out by hand, or found using software such as GAP.

Denote the number of $\text{Aut}(\mathbb{F}_q)$ -orbits of \mathbb{F}_q by Ψ_q . The previous proposition shows that the number of T -systems of $\text{PSL}(2, q)$ is at least equal to $\Psi_q - 1$. The exact number of $\text{Aut}(\mathbb{F}_q)$ -orbits is given by the formula

$$\Psi_q = \frac{1}{s} \sum_{r|s} \varphi(s/r) p^r .$$

Thus, for example, $\text{PSL}(2, 2^{30})$ has at least 35, 792, 567 T -systems.

The formula for Ψ_q can be proven by Möbius inversion, but there is also a very elegant argument using Burnside's Lemma that was shown to us by Gareth Jones.

In further work, we are working to get a better understanding of Nielsen equivalence in $\text{PSL}(2, q)$. The key idea is to consider the function $\mathcal{G}_2(\text{PSL}(2, q)) \rightarrow \mathbb{F}_q^3$ defined by sending (A, B) to $(\text{tr}(A), \text{tr}(B), \text{tr}(AB)) = (\alpha, \beta, \gamma)$. The trace invariant has a fairly simple expression in terms of this map, because the *Fricke-Klein formula* says that

$$\text{tr}([A, B]) = \alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 2 .$$

The Nielsen equivalence classes are the orbits of the right $\text{Aut}(F_2)$ -action on $\mathcal{G}_2(\text{PSL}(2, q))$, and this right action induces an action on \mathbb{F}_q^3 that preserves the level surfaces of the polynomial $\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 2$, since it preserves $\text{tr}([A, B])$. Therefore there is map from the Nielsen classes in $\mathcal{G}_2(\text{PSL}(2, q))$ to the orbits of this action on \mathbb{F}_q^3 . This map on orbits turns out to be 1-to-1 if the characteristic $p = 2$, and (≤ 2) -to-1 (this is essentially a result of Macbeath). If both

- 1) The map on orbits is always 1-to-1, and
- 2) the action is transitive on level surfaces of $\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 2$, (after one excludes the (known) points of \mathbb{F}_q^3 that do not come from generating pairs),

then the number of T -systems is exactly $\Psi_q - 1$. Both 1) and 2) are consistent with our computer calculations for $q \leq 100$.

We mention that W. Goldman has carried out a deep study of the dynamics of the $\text{Aut}(F_2)$ -action on \mathbb{R}^3 determined by $\mathcal{G}_2(\text{PSL}(2, \mathbb{R})) \rightarrow \mathbb{R}^3$, in The modular group action on real $\text{SL}(2)$ -characters of a one-holed torus, *Geom. Topol.* 7 (2003), 443–486. But not much of the characteristic-0 methodology carries over to the case of \mathbb{F}_q .