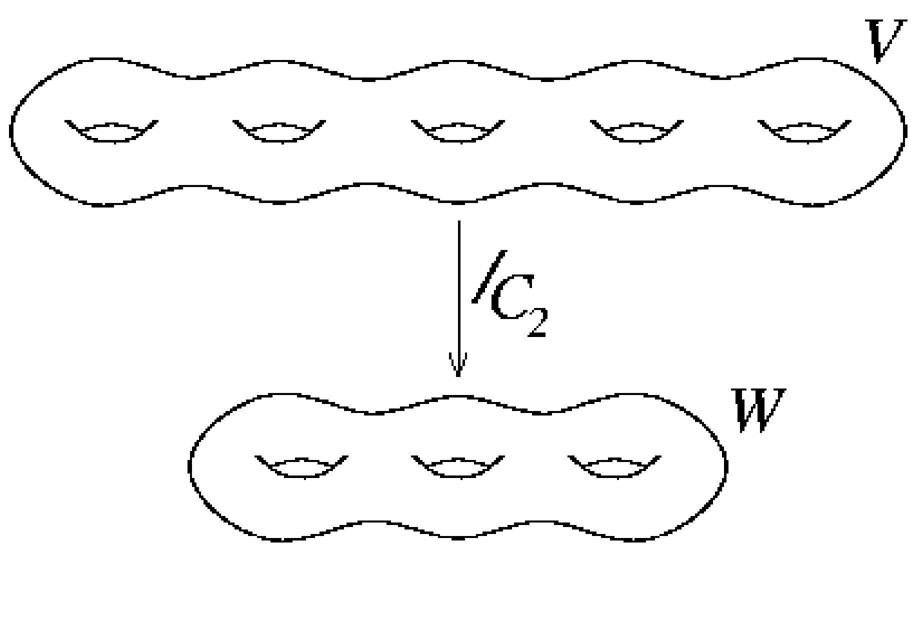
Imbeddings of Free Actions on Handlebodies



handlebody = (compact) 3-dimensional orientable handlebody

action = effective action of a finite group G on a handlebody, by orientation-preserving (smooth- or PL-) homeomorphisms

Actions on handlebodies have been extensively studied. See articles by various combinations of: Bruno Zimmermann, Andy Miller, John Kalliongis, McC.

Free actions on handlebodies have been studied by J. Przytycki, and more recently by McC and M. Wanderley of Universidade Federal de Pernambuco, Brazil.

Elementary Observation: Every finite group acts freely on a handlebody.

Proof: Let W be a handlebody of genus g, where g is at least as large as $\mu(G)$, the minimum number of elements in a generating set for G.

Since $\pi_1(W)$ is free of rank g, there is a surjective homomorphism $\psi\colon \pi_1(W) \to G$.

The covering of W corresponding to the kernel of ψ is a handlebody (since its fundamental group is free), and it admits an action of G as covering transformations, with quotient W. $\square$

But how many different actions are there?

Two actions $\alpha\colon G \to \text{Homeo}(V_1)$ and $\beta\colon G \to \text{Homeo}(V_2)$ are *equivalent* if V_1 and V_2 are equivariantly homeomorphic, that is, there is a homeomorphism $h\colon V_1 \to V_2$ so that $\alpha(\gamma) = h \circ \beta(\gamma) \circ h^{-1}$ for all $\gamma \in G$.

The actions are *weakly equivalent* when they are equivalent after changing one of them by an automorphism of G.

McC-Wanderley proved, among other results, that for every N, there exists a solvable G and a genus g such that G has at least N weak equivalence classes of free actions on the handlebody of genus g (the hard part of this is an algebraic result of M. Dunwoody).

But there is no known counterexample to the following: If G is finite, *and g is greater than the minimum genus* of handlebody on which G can act freely,* then all free actions of G on the genus g handlebody are equivalent.

*The minimum genus is $1 + |G|\ (\mu(G) - 1)$.

A weaker relation than equivalence is that one free G-action imbeds equivariantly in another. This turns out to be a very weak equivalence relation:

Theorem 1 *Let G be a finite group acting freely and preserving orientation on two handlebodies V_1 and V_2, not necessarily of the same genus. Then there is a G-equivariant imbedding of V_1 into V_2.*

In fact, imbedding free actions of handlebodies is ridiculously easy:

Theorem 2 *Let G be a finite group acting freely and preserving orientation on a handlebody V and on a connected 3-manifold X. Then there is a G-equivariant imbedding of V into X.*

1. Since the actions are free and orientation-preserving, V/G is an orientable handlebody W, and X/G is a connected orientable 3-manifold Y.

2. By elementary covering space theory, there are group extensions
$$1 \longrightarrow \pi_1(V) \longrightarrow \pi_1(W) \xrightarrow{\psi} G \longrightarrow 1$$
and
$$1 \longrightarrow \pi_1(X) \longrightarrow \pi_1(Y) \xrightarrow{\Psi} G \longrightarrow 1.$$

3. Regarding W as a regular neighborhood of a graph Γ, choose an imbedding j of Γ into Y so that $\Psi \circ j_{\#} \colon \pi_1(W) = \pi_1(\Gamma) \to \pi_1(Y) \to G$ equals ψ. Since both W and Y are orientable, j extends to an imbedding J of W into Y.

4. The data that $\Psi \circ j = \psi$ translates into the fact that J lifts to a G-equivariant imbedding of V into X.

$$\begin{array}{ccc} V & \longrightarrow & X \\ \big\downarrow /G & & \big\downarrow /G \\ W & \longrightarrow & Y \end{array}$$

One might ask whether, given an action on V, there exists an X for which there is a more "natural" kind of equivariant imbedding— one for which for which V *is one of the handlebodies in a* G*-invariant Heegaard splitting of* X*.*

Simply by forming the double of V and taking an identical action on the second copy of V, one obtains such an extension with X a connected sum of $S^2 \times S^1$'s.

A better question is whether V imbeds as an invariant Heegaard handlebody for a free action on some *irreducible* 3-manifold. Our main result answers this question affirmatively.

Theorem 3 *Any orientation-preserving free* G*-action on a handlebody* V *imbeds equivariantly as a Heegaard handlebody in a free* G*-action on some closed irreducible* 3*-manifold. This* 3*-manifold may be chosen to be Seifert-fibered. Provided that* V *has genus greater than* 1, *it may be chosen to be hyperbolic.*

Here is a sketch of the proof. Again, let $W = V/G$, and $\psi\colon \pi_1(W) \to G$. We will find

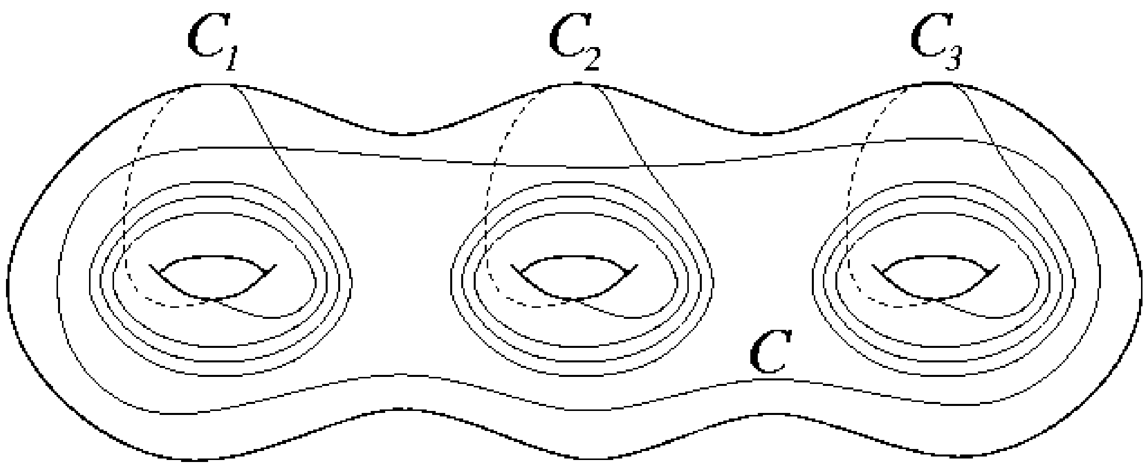
1. an imbedding J of W as a *Heegaard* handlebody in some closed 3-manifold Y, and
2. a homomorphism $\Psi\colon \pi_1(Y) \to G$ with $\Psi \circ J_{\#} = \psi$.

For the lifted imbedding of V into the covering space X of Y, $\overline{X - V}$ is a handlebody, since it covers the handlebody $\overline{Y - W}$. So V imbeds equivariantly as a Heegaard handlebody in X.

To construct Y, we will add g ($=$ genus(W)) 2-handles to W along attaching curves in ∂W, so that

1. The complement in ∂W of the attaching circles is connected. This ensures that the union of W with the 2-handles can be filled in with a 3-ball to make a closed Y that contains W as a Heegaard handlebody.
2. Each 2-handle is attached along a loop in the kernel of ψ. This ensures that $\psi\colon \pi_1(W) \to G$ extends to $\Psi\colon \pi_1(Y) \to G$.

The Seifert-fibered case: Let n be the order of G. Consider the following loops in ∂W, where each C_i goes n times around one of the handles of W.

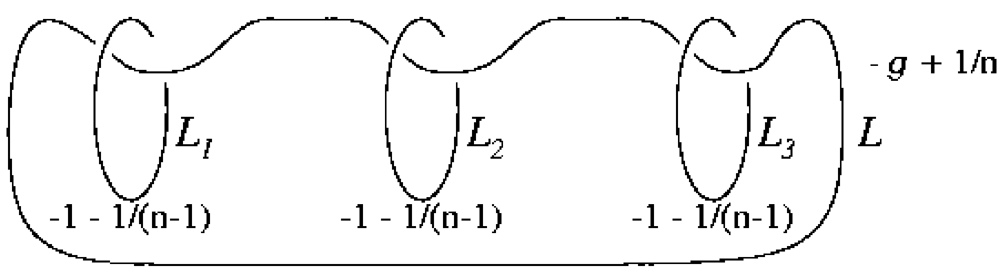


Let C_i' be the images of the C_i under the n^{th} power of a Dehn twist of ∂W about C. These C_i' are the attaching curves for the 2-handles. The complement of $\cup C_i$ is connected, so the complement of $\cup C_i'$ is also connected.

If $x_1, \ldots,\ x_g$ are a standard set of generators of $\pi_1(W)$, where x_i goes once around the i^{th} 1-handle of W, then C_i represents x_i^n (up to conjugacy), and C_i' represents $x_i^n\,(x_1 \cdots x_g)^{-n}$. So ψ carries each C_i' to the trivial element of G, and ψ induces $\Psi\colon \pi_1(Y) \to G$.

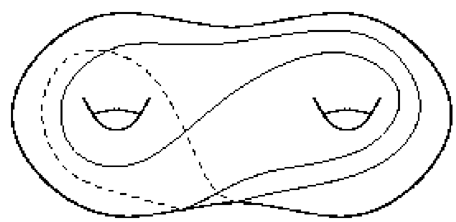
By a construction that goes back (at least) to Lickorish's proof that all closed orientable 3-manifolds are cobordant to the 3-sphere, we may change the attaching map of a Heegard splitting, at the expense of introducing Dehn surgeries on solid tori imbedded in one of the Heegard handlebodies.

We do this to the previous Heegaard description, to move the C_i' to a standard set of attaching curves for S^3. This yields the following surgery description of Y:

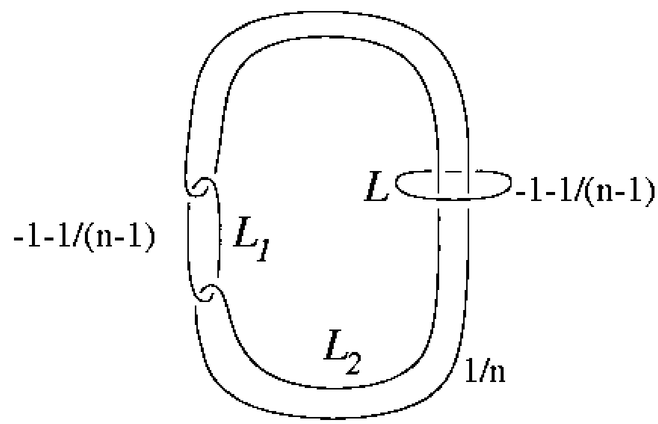


The complement of this link in S^3 is just a g-times punctured disc times S^1, which has a product S^1-fibering. The core circles of the filled-in solid tori become exceptional Seifert fibers. The Seifert invariants of Y work out to be $\{-1; (o_1, 0); (n, 1), \ldots, (n, 1), (n, n-1)\}$.

The hyperbolic case: Suppose for now that the genus of W is 2. Take the same curves C_1 and C_2 as before, but instead of the curve C used before, use the curve shown here:

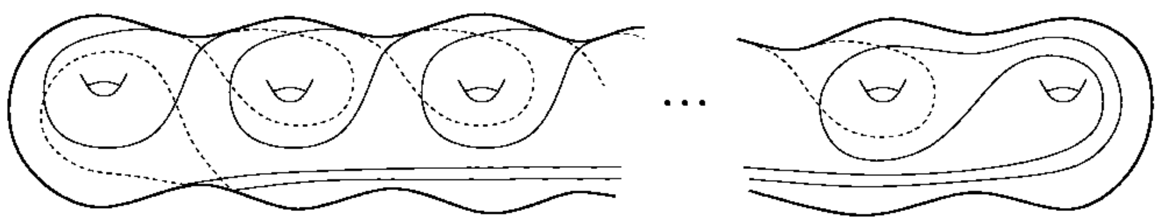


It turns out that the resulting surgery description for Y is:



The link complement is a 2-fold cover of the Whitehead link complement, so is hyperbolic. Conceivably, this Dehn filling does not produce a hyperbolic 3-manifold, but n can be any integer divisible by the order of G, and all but finitely many choices yield a hyperbolic Y.

If the genus of W is g, then in place of C we use the following collection of curves:



The resulting surgery description of Y is similar, but instead of a two-component chain linking the loop L, we obtain a $(2g-2)$-component chain. The complement is a $(2g-2)$-fold cover of the Whitehead link, so is hyperbolic.

Again, the surgery coefficients are simple expressions in n, and all but finitely many choices for n must yield a hyperbolic Y.