ORIENTATION-REVERSING FREE ACTIONS ON HANDLEBODIES

ANTONIO F. COSTA AND DARRYL MCCULLOUGH

Abstract. We examine free orientation-reversing group actions on orientable handlebodies, and free actions on nonorientable handlebodies. A classification theorem is obtained, giving the equivalence classes and weak equivalence classes of free actions in terms of algebraic invariants that involve Nielsen equivalence. This is applied to describe the sets of free actions in various cases, including a complete classification for many (and conjecturally all) cases above the minimum genus. For abelian groups, the free actions are classified for all genera.

The orientation-preserving free actions of a finite group G on 3-dimensional orientable handlebodies have a close connection with a long-studied concept from group theory, namely *Nielsen equivalence* of generating sets. The basic result is that the orientation-preserving free actions of G on the handlebody of genus g, up to equivalence, correspond to the Nielsen equivalence classes of n-element generating sets of G, where $n = 1 + (g-1)/|G|$. This has been known for a long time; it is implicit in work of J. Kalliongis and A. Miller in the 1980's, as a direct consequence of theorem 1.3 in their paper [7] (for free actions, the graph of groups will have trivial vertex and edge groups, and the equivalence of graphs of groups defined there is readily seen to be the same as Nielsen equivalence on generating sets of G). As far as we know, the first explicit statement detailing the correspondence appears in [13], which also contains various applications and calculations using it.

In this paper, we extend the theory from [13] to free actions that contain orientation-reversing elements, and to free actions on nonorientable handlebodies. The orbits of a certain group action on the collection $\mathcal{G}_n$ of n-element generating sets are the Nielsen equivalence classes, and this action extends to an action on a set $\mathcal{G}_n \times \mathbb{V}_n$, in such a way that the orbits correspond to the equivalence classes of all free G-actions on handlebodies of genus $1+(n-1)|G|$. This correspondence is given as theorem 1.1, which is proven in section 4 after presentation of preliminary material on Nielsen equivalence in section 2, and on "uniform homeomorphisms" in section 3. From

Date: January 30, 2005.

2000 *Mathematics Subject Classification.* Primary 57M60; Secondary 20F05.

Key words and phrases. 3-manifold, handlebody, group action, free, orientation-reversing, nonorientable, Nielsen, Nielsen equivalence, abelian.

The first author was supported in part by BFM 2002-04801, the second author was supported in part by NSF grant DMS-0102463.

theorem 1.1, more specific results are derived in section 5 for orientation-reversing free actions on orientable handlebodies, and in section 6 for free actions on nonorientable handlebodies. These are illustrated by several calculations for specific groups, and in section 7 we use the results to classify all free actions of abelian groups on handlebodies, extending the classification of orientation-preserving actions given in [13].

We should mention that nonfree actions on handlebodies have been examined in considerable depth. For nonfree actions, the natural structure on the quotient object is that of an orbifold, rather than just a handlebody, and the resulting analysis is much more complicated. A general theory of actions was given in [12] and the aforementioned [7], and the actions on very low genera were extensively studied in [8]. Actions with the genus small relative to the order of the group were investigated in [14] and [17], and the special case of orientation-reversing involutions is treated in [6]. The first focus on free actions seems to be [16], which examines free actions of the cyclic group.

The authors gratefully acknowledge the support of the U. S. National Science Foundation, the Ministerio de Educación y Ciencia of Spain, and the Mathematical Research and Conference Center in Będlewo of the Polish Academy of Sciences.

1. The classification theorem

In this paper, G will always denote a finite group. A *G-action* on a space X is an injective homomorphism $\Phi\colon G \to \mathrm{Homeo}(X)$. Two actions $\Phi_1, \Phi_2\colon G \to \mathrm{Homeo}(X)$ are said to be *equivalent* if they are conjugate as representations, that is, if there is a homeomorphism $h\colon X \to X$ such that $h\Phi_1(g)h^{-1} = \Phi_2(g)$ for each $g \in G$. They are *weakly equivalent* if their images are conjugate, that is, if there is a homeomorphism $h\colon X \to X$ so that $h\Phi_1(G)h^{-1} = \Phi_2(G)$. Equivalently, there is some automorphism α of G so that $h\Phi_1(g)h^{-1} = \Phi_2(\alpha(g))$ for all g. In words, equivalent actions are the same after a change of coordinates on the space, while weakly equivalent actions are the same after a change of coordinates on the space and a change of the group by automorphism. If X is homeomorphic to Y, then the sets of equivalence (or weak equivalence) classes of actions on X and on Y can be put into correspondence using any homeomorphism from X to Y.

To state the classification theorem, we first fix a basis $x_1, \ldots, x_n$ of F_n (where F_n is the free group of rank n). Such a selection gives an identification of the direct product G^n with the set $\mathrm{Hom}(F_n, G)$ of group homomorphisms from F_n to G, by regarding $(g_1, \ldots, g_n)$ as the homomorphism $\gamma(g_1, \ldots, g_n)\colon F_n \to G$ that sends x_i to g_i. An action of $\mathrm{Aut}(F_n) \times \mathrm{Aut}(G)$ on G^n is then defined by $(\phi, \alpha) \cdot \gamma = \alpha \circ \gamma \circ \phi^{-1}$. Now, write $\mathbb{V}_n$ for the direct sum $\oplus_{i=1}^{n} C_2$, where $C_2 = \{-1, 1\}$. Using the selected basis $x_1, \ldots, x_n$ of F_n, identify $\mathbb{V}_n$ with $\mathrm{Hom}(F_n, C_2)$ by identifying an element $(v_1, \ldots, v_n)$ of $\mathbb{V}_n$ with the homomorphism $\omega(v_1, \ldots, v_n)$ that sends x_i to v_i. We define an $\mathrm{Aut}(F_n) \times \mathrm{Aut}(G)$-action on $G^n \times \mathbb{V}_n$ by putting

$$(\phi, \alpha) \cdot (\gamma, \omega) = (\alpha \circ \gamma \circ \phi^{-1}, \omega \circ \phi^{-1}) \ .$$

The elements of the set $\mathcal{G}_n$ of generating n-vectors of G correspond to the surjective elements of $\mathrm{Hom}(F_n, G)$, so $\mathcal{G}_n$ and $\mathcal{G}_n \times \mathbb{V}_n$ are invariant under the $\mathrm{Aut}(F_n)$- and $\mathrm{Aut}(F_n) \times \mathrm{Aut}(G)$-actions respectively.

Under the action of $\mathrm{Aut}(F_n)$ (or $\mathrm{Aut}(F_n) \times \mathrm{Aut}(G)$) on $\mathbb{V}_n$ the element $(1, \ldots, 1)$ is fixed, so the subset $\mathcal{G}_n \times \{(1, \ldots, 1)\}$ is a union of orbits. Restricted to the subset $G^n \times \{(1, \ldots, 1)\}$, the $\mathrm{Aut}(F_n) \times \mathrm{Aut}(G)$-action can be identified with the action originally defined on G^n.

In section 3 we will define a collection of handlebodies $N(v_1, \ldots, v_n)$ of genus n, one for each element of $\mathbb{V}_n$. It includes both orientable and nonorientable handlebodies. Given a free action Φ of G on a handlebody V, with quotient a handlebody N of genus n, choose any $N(v_1, \ldots, v_n)$ that is homeomorphic to N, and fix a homeomorphism $k\colon N \to N(v_1, \ldots, v_n)$. Let W be the covering of $N(v_1, \ldots, v_n)$ determined by the subgroup $k_\#(\pi_1(V))$, where $k_\#$ is the isomorphism induced by k on the fundamental groups. A lift of k to a homeomorphism from V to W identifies G with the group of covering transformations of W. The free group $F_n = \pi_1(N(v_1, \ldots, v_n))$ has a basis $x_1, \ldots,\ x_n$ (defined in section 3). Each x_i determines a covering transformation $g_i \in G$. We associate to Φ the pair $((g_1, \ldots, g_n), (v_1, \ldots, v_n))$, which we will abbreviate as (g, v). Since the x_i generate F_n, the g_i generate G, so (g, v) is an element of $\mathcal{G}_n \times \mathbb{V}_n$, where $\mathcal{G}_n$ denotes the elements of G^n whose elements form a generating set. The following theorem gives a complete algebraic classification of free actions on orientable and nonorientable handlebodies.

Theorem 1.1. *Sending Φ to the orbit of the element (g, v) defines a bijection from the equivalence classes (respectively, weak equivalence classes) of free G-actions on handlebodies of genus $1 + |G|\,(n-1)$ to the set of Aut(F_n)-orbits (respectively, Aut$(F_n) \times$ Aut(G)-orbits) in $\mathcal{G}_n \times \mathbb{V}_n$.*

We call a G-action on a handlebody V *orientation-preserving* if V is orientable and each element of G acts preserving orientation, and *orientation-reversing* if V is orientable and some element of G acts reversing orientation. We will see that the orbits contained in $G^n \times \{(1, \ldots, 1)\}$ correspond exactly to the equivalence classes (or weak equivalence classes) of orientation-preserving free actions. This recovers the algebraic classification of orientation-preserving actions given in [13, Theorem 2.3]. Corollary 5.2 characterizes the orientation-reversing actions in the context of theorem 1.1. In section 6 we shall apply theorem 1.1 to the classification of actions on nonorientable handlebodies.

As mentioned in the introduction, theorem 1.1 is proven in section 4 after presentation of preliminary material in sections 2 and 3. We should mention that section 2, which presents the $\mathrm{Aut}(F_n) \times \mathrm{Aut}(G)$-action on G^n defined above in terms of the classical notion of Nielsen equivalence, is not absolutely essential to our work here. But it would be peculiar indeed to omit this interpretation from our exposition, and moreover the language of Nielsen equivalence is used in [13], so the interpretation is needed to clarify how our present work recovers the orientation-preserving case.

From now on, the term *action* will mean a *free* action of a finite group on a 3-dimensional handlebody V_g of genus $g \geq 1$ (only the trivial group can act freely on the handlebody of genus 0, the 3-ball). One may work in either of the categories of piecewise-linear or smooth actions. We assume that one of these two categories has been chosen, and that all maps, isotopies, etc. lie in that category.

2. Nielsen equivalence

It will be convenient to define Nielsen equivalence in terms of group actions on sets. We write C_k for the cyclic group of order $k \geq 2$, including the infinite cyclic group C_∞. Let $\mathbb{U} \cong C_2 * C_2 * C_2 * C_\infty$ be given by the presentation

$$\mathbb{U} = \langle t, u, v, w \mid t^2 = u^2 = v^2 = 1 \rangle.$$

For any group G and any positive integer $n \geq 2$, an action of $\mathbb{U}$ on the n-fold direct product G^n is defined by

$$\begin{aligned} t(g_1, g_2, \ldots, g_n) &= (g_1^{-1}, g_2, \ldots, g_n) \\ u(g_1, g_2, g_3, \ldots, g_n) &= (g_1^{-1}, g_1 g_2, \ldots, g_n) \\ v(g_1, g_2, g_3, \ldots, g_n) &= (g_2, g_1, g_3, \ldots, g_n) \\ w(g_1, g_2, \ldots, g_n) &= (g_n, g_1, g_2, \ldots, g_{n-1}) \ . \end{aligned}$$

The orbits of this $\mathbb{U}$-action on G^n are called *Nielsen equivalence classes.*

Note that if the elements of two Nielsen equivalent n-tuples are regarded as subsets of G, then they generate the same subgroup of G. In particular, if the entries of one of them generate G, the same is true for the other.

Conjugates of t by w allow one to replace any g_i by its inverse. Conjugates of v by w allow one to interchange any g_i with any g_{i+1}, and hence to effect any permutation of the coordinates. Simple combinations of these with u allow one to replace any g_i by $g_i g_j^{\pm 1}$ or $g_j^{\pm 1} g_i$ for some $j \neq i$, keeping all other coordinates fixed. On the other hand, each of the four generators results from some sequence of these basic Nielsen "moves". Thus Nielsen equivalence is often described as the equivalence relation generated by these basic moves.

By letting $\mathrm{Aut}(G)$ act on the left of G^n coordinatewise, we can extend the $\mathbb{U}$-action to a $\mathbb{U} \times \mathrm{Aut}(G)$-action. This adds the additional basic Nielsen move

$$\alpha(g_1, \ldots, g_n) = (\alpha(g_1), \ldots, \alpha(g_n))$$

for any $\alpha \in \mathrm{Aut}(G)$. The orbits of this $\mathbb{U} \times \mathrm{Aut}(G)$-action are called *weak Nielsen equivalence classes.*

The next lemma shows that the action of $\mathbb{U}\times\mathrm{Aut}(G)$ on G^n always factors through the action of $\mathrm{Aut}(F_n)\times\mathrm{Aut}(G)$ on G^n that we defined in section 1. Consequently, the Nielsen equivalence classes in G^n are exactly the orbits of this action.

Lemma 2.1. *The orbits of the $Aut(F_n)$-action on G^n (respectively, the $Aut(F_n)\times Aut(G)$-action on G^n) are exactly the Nielsen equivalence classes (respectively, the weak Nielsen equivalence classes). In fact, there is a surjective homomorphism $A_n\colon \mathbb{U}\to Aut(F_n)$ such that the action of an element $(u,\alpha)\in\mathbb{U}\times Aut(G)$ equals the action of $(A_n(u),\alpha)$. Changing the choice of basis for F_n changes A_n by an inner automorphism of $Aut(F_n)$.*

Proof. Define $T\in\mathrm{Aut}(F_n)$ by $T(x_1)=x_1^{-1}$ and $T_i(x_j)=x_j$ for $j>1$, and similarly define U, V, and W corresponding to u, v, and w. It is straightforward to check that $(t,\alpha)(g_1,\ldots,g_n)=(T,\alpha)(g_1,\ldots,g_n)$, and similarly for the other three generators, so the action of $\mathbb{U}$ on G^n factors through the image of the “capitalization” function $A_n\colon\mathbb{U}\to\mathrm{Aut}(F_n)$. Using well-known generating sets for $\mathrm{Aut}(F_n)$, such as that of Nielsen's presentation [15] or the Fouxe-Rabinovitch presentation listed in [11], one checks that A_n is surjective. The basis change remark is a straightforward check. □

The $\mathrm{Aut}(F_n)\times\mathrm{Aut}(G)$-action on $G^n\times\mathbb{V}_n$ can be regarded as extending the definition of Nielsen equivalence in G^n to the set $G^n\times\mathbb{V}_n$. In the next section, we will see how this extended equivalence will capture some orientation information when we apply it to study actions on handlebodies.

3. Uniform homeomorphisms

We will use an idea which has appeared several times in the literature [1], [10], [11] (the most relevant of these references is [11], since it also concerns handlebodies). The quotient of a free action on a genus g handlebody is a handlebody V_n of genus $n=1+(g-1)/|G|$ (see section 4). This handlebody is regarded as one component of a disjoint union of a family of handlebodies indexed by $\mathbb{V}_n$, where the handlebody $N(v_1,\ldots,v_n)$ corresponding to a vector $(v_1,\ldots,v_n)$ has the property that traveling around the i^{th} handle reverses the local orientation exactly when $v_i=-1$. An n-tuple $(g_1,\ldots,g_n)$ of elements that generate G determines a G-action on a handlebody with quotient $N(v_1,\ldots,v_n)$ in the following way: G acts by covering transformations on the covering space of $N(v_1,\ldots,v_n)$ corresponding to the kernel of the homomorphism $\pi_1(N(v_1,\ldots,v_n))\to G$ that sends the generator corresponding to the i^{th} handle to g_i.

A key property of this family of handlebodies is that any element of $\mathrm{Aut}(\pi_1(V_n))$ can be realized, in an appropriate sense, by a “uniform” homeomorphism of the family. The action of uniform homeomorphisms on the set of components of the family corresponds exactly to the $\mathrm{Aut}(F_n)$-action on $\mathbb{V}_n$ defined in section 2. Uniform homeomorphisms overcome the technical problem that an automorphism of $\pi_1(V_n)$ need not preserve the orientability of 1-handles and hence need not be induced by a self-homeomorphism of V_n.

The proof of the main technical result, theorem 1.1, shows that two pairs $((g_1,\ldots,g_n),(v_1,\ldots,v_n))$ and $((g_1',\ldots,g_n'),(v_1',\ldots,v_n'))$ in $\mathcal{G}_n\times\mathbb{V}_n$ lie in the same $\mathrm{Aut}(F_n)$-orbit exactly when there is a homeomorphism between $N(v_1,\ldots,v_n)$ and $N(v_1',\ldots,v_n')$ that lifts to an equivalence between the actions which have them as quotients and are determined by $(g_1,\ldots,g_n)$ and $(g_1',\ldots,g_n')$.

Here is the construction from [11]. Fixing a positive integer n, let R_n be a 1-point union of n circles. Write F_n for the free group $\pi_1(R_n)$. Let $x_1,\ldots,$ x_n be the standard set of generators of F_n, where x_i is represented by a loop that travels once around the i^{th} circle.

To set notation, let Σ be a 3-ball, and in $\partial\Sigma$ select $2n$ disjoint imbedded 2-disks $D_1,E_1,D_2,E_2,\ldots,D_n,E_n$. Fix orientation-preserving imbeddings $J_i\colon D^2\to D_i$ and $K_i\colon D^2\to E_i$. Let $r\colon D^2\to D^2$ send (x,y) to $(x,-y)$. For $v=(v_1,\ldots,v_n)\in\mathbb{V}_n$, construct a handlebody $N(v)$ as follows. For each i, let H_i be a copy of $D^2\times I$ and identify $(x,y,0)$ with $J_i(x,y)$ and $(x,y,1)$ with $K_ir^{(1+v_i)/2}(x,y)$. The resulting 1-handle H_i is orientation-preserving or orientation-reversing according as v_i is 1 or -1.

Regard $N(v)$ as a thickening of R_n, in which the join point is the center $*$ of Σ and the loop of R_n that represents x_i goes once over H_i from D_i to E_i and does not meet any other H_j. Traveling around this i^{th} loop preserves the local orientation at $*$ if and only $v_i=1$. Thus $N(1,\ldots,1)$ is orientable, while all other $N(v)$ are nonorientable and are homeomorphic to $N(-1,\ldots,-1)$. We denote the disjoint union of the $N(v)$ by $\mathcal{N}$.

We will now define a homeomorphism of $\mathcal{N}$ called a uniform slide homeomorphism. For each $N(v)$, write $N'(v)$ for the closure of $N(v)-H_1$. Choose a loop α in $\partial N'(v)$, based at the origin in E_1, that travels through $\partial\Sigma$ to ∂E_2, once over H_2 to ∂D_2, and returns in $\partial\Sigma$ to the origin of E_1. There is an isotopy J_t of $N'(v)$ such that

(1) J_0 is the identity of $N'(v)$,
(2) each J_t the identity outside a regular neighborhood of $E_1\cup\alpha$,
(3) during J_t, E_1 moves once around α, traveling over H_2 from E_2 to D_2, and
(4) the restriction of J_1 to E_1 is the identity or r, according as J_1 preserves or reverses the local orientation on E_1.

A homeomorphism of $\mathcal{N}$ is defined by sending $N(v)$ to $N(w)$ using J_1 on $N'(v)$ and the identity on H_1. Here, $(w_1,\ldots,w_n)=(v_1v_2,v_2,\ldots,v_n)$, since the r in item (4) will be needed exactly when $v_2=-1$. There are many choices of sliding loop α, nonisotopic in $\partial N'(v)$, so the homeomorphism of $\mathcal{N}$ is by no means uniquely defined up to isotopy.

With respect to the identifications $\pi_1(R_n)=\pi_1(N(v))$ given by the inclusions of R_n into $N(v)$ and $N(w)$, the homeomorphism from $N(v)$ to $N(w)$ induces the automorphism ρ of F_n that sends x_1 to x_1x_2 and fixes all other

x_j. Note that $(w_1, w_2, \ldots, w_n) = \rho \cdot (v_1, v_2, \ldots, v_n)$, for the action of ρ on $(v_1, v_2, \ldots, v_n)$ defined in section 2.

This particular basic slide homeomorphism is called sliding the right end (that is, E_1) of H_1 over H_2. Similarly, one can uniformly slide the right or left end of any H_i over any other H_j, either from E_j to D_j or from D_j to E_j, obtaining homeomorphisms whose effect on components of $\mathcal{N}$ agrees with the action of their induced automorphisms on $\mathbb{V}_n$. These are called *uniform slide homeomorphisms* of $\mathcal{N}$.

A *uniform interchange* of H_i and H_j is defined using an isotopy J_t that interchanges both D_i and D_j, and E_i and E_j. It sends $N(\ldots, v_i, \ldots, v_j, \ldots)$ to $N(\ldots, v_j, \ldots, v_i, \ldots)$, and induces the automorphism of F_n that interchanges x_i and x_j. Using a J_t that interchanges D_i and E_i defines a *uniform spin* of the i^{th} handle. This preserves each component of $\mathcal{N}$, and induces the automorphism that sends x_i to x_i^{-1}.

There are two other kinds of basic uniform homeomorphisms, both of which preserve each $N(v)$ and induce the identity automorphism on F_n. Choose a reflection of Σ that preserves $*$ and restricts to r on each D_i and E_i. Define a homeomorphism of $N(v)$ by taking $r \times 1_I$ on each H_i and the chosen reflection on Σ. The resulting uniform homeomorphism of $\mathcal{N}$ is denoted by R. Finally, any Dehn twist about a properly imbedded 2-disk in $\mathcal{N}$ is a basic uniform homeomorphism.

In all cases, the action of the basic uniform homeomorphism on the components of $\mathcal{N}$ agrees with the action on $\mathbb{V}_n$ of the automorphism it induces on F_n with respect to the identifications $F_n = \pi_1(R_n) = \pi_1(N(v))$.

A *uniform homeomorphism* of $\mathcal{N}$ is a homeomorphism (freely) isotopic to a composition of the basic uniform homeomorphisms we have defined here. The inverse of a basic uniform homeomorphism is a basic uniform homeomorphism, so the inverse of any uniform homeomorphism is uniform.

By abuse of notation, we write $*$ for the union of the basepoints of the components of $\mathcal{N}$, and by $\mathcal{M}(\mathcal{N}, *)$ the group of isotopy classes of homeomorphisms of $\mathcal{N}$ that preserve this subset. The uniform homeomorphisms that preserve $*$ form a subgroup $\mathcal{U}(\mathcal{N}, *)$ of $\mathcal{M}(\mathcal{N}, *)$, called the *uniform mapping class group.* We mention that although we have given infinitely many generators, it can be shown that $\mathcal{U}(\mathcal{N}, *)$ is finitely generated. This is proven in [11].

For $v \in \mathbb{V}_n$, let $\mathrm{St}(N(v), *) \subseteq \mathcal{U}(\mathcal{N}, *)$ be the stabilizer of the component $N(v)$ under the action of $\mathcal{U}(\mathcal{N}, *)$ on the components of $\mathcal{N}$. We have the following result from [11]:

Theorem 3.1. *The restriction $St(N(v), *) \to \mathcal{M}(N(v), *)$ is surjective. Any homeomorphism $N(v) \to N(w)$ is isotopic to the restriction of a uniform homeomorphism.*

Proof. The first statement is basically theorem 7.2.3 from [11], proven there for compression bodies, which include handlebodies as a special case. The restriction in [11] to mapping classes of local degree 1 at $*$ is not needed since we have included the reflection R among our uniform homeomorphisms. For the second statement, note first that the uniform homeomorphisms act transitively on the set of nonorientable components of $\mathcal{N}$, so given $g\colon N(v) \to N(w)$, there is a uniform homeomorphism u_1 that carries $N(w)$ to $N(v)$. (To see this, suppose that $N(w)$ and $N(v)$ are nonorientable and choose some $w_i = -1$. Slide the other handles of $N(w)$ over the i^{th} handle as necessary to make $w_j = v_j$ for $j \neq i$. If all these w_j are now 1, then $w_i = -1 = v_i$ since $N(w)$ and $N(v)$ are nonorientable. If not, there is some other $w_j = -1$, and a slide of the i^{th} handle over the j^{th} can be used if needed to change w_i to equal v_i.) By the first sentence of the theorem, the composition $u_1 \circ g$ is isotopic to the restriction of a uniform homeomorphism u_2 that stabilizes $N(v)$, so on $N(v)$, g is isotopic to $u_1^{-1} \circ u_2$. □

4. The algebraic classification of actions

Suppose that $\Phi\colon G \to \mathrm{Homeo}(V)$ is a free action on a handlebody V, possibly nonorientable. Its quotient N is also a handlebody. To see this, recall that any torsionfree finite extension of a finitely generated free group is free (by [9] any finitely generated virtually free group is the fundamental group of a graph of groups with finite vertex groups, and if the group is torsionfree, the vertex groups must be trivial), so $\pi_1(V/G)$ is free. Since V is irreducible, so is V/G, and theorem 5.2 of [5] shows that V/G is a handlebody.

From covering space theory, the action Φ determines an extension

$$1 \longrightarrow \pi_1(V) \longrightarrow \pi_1(N) \xrightarrow{\pi} G \longrightarrow 1$$

where $\pi(x)$ is defined by taking a representative loop for x, lifting it to a path starting at the basepoint of V, and letting $\pi(x)$ be the covering transformation that sends the basepoint of V to the endpoint of the path. Writing n for the genus of N, the Euler characteristic shows that $1+|G|\,(n-1)$ is the genus of V. The genus of N can be any n greater than or equal to $\mu(G)$, the minimum number of elements in a generating set of G. In particular, the genera of handlebodies on which G acts freely preserving orientation are exactly $1 + |G|(n-1)$ where $n \geq \mu(G)$. The *minimal genus* is $1 + |G|(\mu(G) - 1)$.

In the remainder of this section, we will prove theorem 1.1. Recall that in section 1, we associated to an action Φ of G on a handlebody V an element $(g, v) = ((g_1, \ldots, g_n), (v_1, \ldots, v_n))$ of $\mathcal{G}_n \times \mathbb{V}_n$. It was defined by taking a homeomorphism k from the quotient handlebody V/G to some $N(v_1, \ldots, v_n)$, letting W be the covering of $N(v_1, \ldots, v_n)$ determined by the subgroup $k_\#(\pi_1(V))$, and putting g_i equal to the lift of the element $x_i \in \pi_1(N(v_1, \ldots, v_n))$ to a covering transformation g_i of W.

First we address the issues of well-definedness. The subgroup $k_\#(\pi_1(V))$ is well-defined up to conjugacy, so W depends only on the choice of k. Changing the choice of basepoint in W or the lift of k changes $((g_1, \ldots, g_n), v)$ to $((hg_1h^{-1}, \ldots, hg_nh^{-1}), v)$ for some $h \in G$. Choose an element $\widetilde{h} \in F_n$ with

$\gamma(g_1,\ldots,g_n)(\widetilde{h}) = h$, and let $\mu(\widetilde{h}) \in \mathrm{Aut}(F_n)$ be the automorphism that conjugates by $\widetilde{h}^{-1}$. Then $((hg_1h^{-1},\ldots,hg_nh^{-1}),v) = (\mu(\widetilde{h}),1)\cdot((g_1,\ldots,g_n),v)$, so these elements lie in the same $\mathrm{Aut}(F_n)$-orbit.

Suppose a different $N(v')$ and $k'\colon N \to N(v')$ are used to associate a pair $(g',v') = ((g_1',\ldots,g_n'),(v_1',\ldots,v_n'))$ to Φ. By theorem 3.1, $k' \circ k^{-1}\colon N(v) \to N(v')$ is the restriction of a uniform homeomorphism u. We claim that $(u_\#,1) \in \mathrm{Aut}(F_n)$ carries (g,v) to (g',v'). Since the action of $\mathcal{U}(\mathcal{N},*)$ on the components of $\mathcal{N}$ induces the action of $\mathrm{Aut}(F_n)$ on $\mathbb{V}_n$, it suffices to show that $\gamma(g_1,\ldots,g_n)\circ u_\#^{-1} = \gamma(g_1',\ldots,g_n')$, that is, that $g_i' = \gamma(g_1,\ldots,g_n)(u_\#^{-1}(x_i))$.

Let (W,w) and (W',w') be the covering spaces of $N(v_1,\ldots,v_n)$ and $N(v_1',\ldots,v_n')$ respectively, such that lifting x_i to W and W' produces g_i and g_i' respectively. Let $\widetilde{u}\colon (W,w) \to (W',w')$ be the lift of $u|_{N(v)}$. Now, g_i' is the covering transformation that carries w' to the endpoint of the lift of x_i starting at w'. Consider $(u|_{N(v)})^{-1}(x_i)$. Its lift to W starting at w is carried by $\widetilde{u}$ to the lift of x_i in W' starting at w'. That is, the covering transformation of W corresponding to g_i' under $\widetilde{u}$ is determined by $u_\#^{-1}(x_i)$, so is $\gamma(g_1,\ldots,g_n)(u_\#^{-1}(x_i))$. This verifies the claim.

Equivalent actions produce equivalent associated elements. For if Φ is equivalent to another G-action Φ' on V', with quotient N', then there is a homeomorphism $j\colon N' \to N$ that lifts to an equivariant homeomorphism from V' to V. Since we may use $k \circ j$ as the homeomorphism from N' to $N(v)$ to define the element associated to Φ', the associated pairs are in the same $\mathrm{Aut}(F_n)$-orbit.

Conversely, suppose that the pairs (g,v) and (g',v') associated to the actions Φ and Φ' are in the same $\mathrm{Aut}(F_n)$-orbit. Let $\phi \in \mathrm{Aut}(F_n)$ carry one to the other. By theorem 3.1, there is a uniform homeomorphism $u \in \mathcal{U}(\mathcal{N})$ inducing ϕ, which must carry $N(v_1,\ldots,v_n)$ to $N(v_1',\ldots,v_n')$. The condition that $\gamma\circ\phi^{-1} = \gamma'$ ensures that u lifts to a G-equivariant homeomorphism from (W,w) to (W',w'), so the actions on these covering spaces are equivalent. Since the actions on W and W' are respectively equivalent to the original actions on V and V', the original actions were equivalent.

Finally, being able to apply an automorphism of G at any point in the process changes equivalence to weak equivalence, and enlarges the choices of (g,v) to the $\mathrm{Aut}(F_n)\times\mathrm{Aut}(G)$-orbit.

5. Actions on orientable handlebodies

From theorem 1.1, an explicit representative of the equivalence class of G-actions corresponding to the $\mathrm{Aut}(F_n)$-orbit of the element (g,v) of $\mathcal{G}_n \times \mathbb{V}_n$ is the covering space W of $N(v)$ whose fundamental group is the kernel of $\gamma = \gamma(g_1,\ldots,g_n)\colon F_n \to G$. Since v_i tells the orientability of x_i in $N(v)$, a covering space is orientable if and only if it corresponds to a subgroup in the kernel of $\omega = \omega(v_1,\ldots,v_n)$. Therefore there is a simple criterion for W to be orientable:

Proposition 5.1. *Let W be the covering space of $N(v)$ corresponding to the kernel of γ. Then W is orientable if and only if there is $\overline{\omega} \in Hom(G, C_2)$ such that $\omega\colon F_n \to C_2$ factors as $\overline{\omega} \circ \gamma\colon F_n \to G \to C_2$. Equivalently, sending g_i to v_i defines a homomorphism from G to C_2.*

Applying theorem 1.1, we obtain:

Corollary 5.2. *Under the correspondence of theorem 1.1, the equivalence classes (respectively, weak equivalence classes) of free G-actions on orientable handlebodies of genus $1+|G|\,(n-1)$ correspond to the set of $Aut(F_n)$-orbits (respectively, $Aut(F_n) \times Aut(G)$-orbits) in $\mathcal{G}_n \times \mathbb{V}_n$ for which sending g_i to v_i (on one, hence on any representative) determines a homomorphism $\overline{\omega}$ from G to C_2.*

It will be useful to make explicit the induced action of $\mathrm{Aut}(F_n) \times \mathrm{Aut}(G)$ on these $\overline{\omega}$. In the statement of proposition 5.3, we call $\overline{\omega}$ the element of $\mathrm{Hom}(G, C_2)$ *associated to* (g, v).

Proposition 5.3. *If $\overline{\omega} \in Hom(G, C_2)$ is associated to $(g, v) \in \mathcal{G}_n \times \mathbb{V}_n$ and $(\phi, \alpha) \in Aut(F_n) \times Aut(G)$, then $\overline{\omega} \circ \alpha^{-1}$ is the element of $Hom(G, C_2)$ associated to $(\phi, \alpha) \cdot (g, v)$.*

Proof. Regarding (g, v) as (γ, ω) we have $(\phi, \alpha) \cdot (\gamma, \omega) = (\alpha \circ \gamma \circ \phi^{-1}, \omega \circ \phi^{-1})$. Since $\omega \circ \phi^{-1} = (\overline{\omega} \circ \alpha^{-1}) \circ (\alpha \circ \gamma \circ \phi^{-1})$, its associated element is $\overline{\omega} \circ \alpha^{-1}$. $\square$

The classification up to equivalence of free actions on orientable handlebodies is no more difficult than the classification of generating n-vectors of G up to Nielsen equivalence. For $n \geq \mu(G)$ let $\mathcal{E}_n$ denote the set of Nielsen equivalence classes of generating n-vectors of G. We write $\mathrm{Epi}(G, C_2)$ for the set of surjective homomorphisms from G to C_2, that is, all elements of $\mathrm{Hom}(G, C_2)$ except the trivial homomorphism 0.

Theorem 5.4. *For $n \geq \mu(G)$, the set of equivalence classes of free G-actions on the orientable handlebody of genus $1 + |G|(n-1)$ corresponds bijectively to $\mathcal{E}_n \times Hom(G, C_2)$, with the orientation-preserving actions corresponding to $\mathcal{E}_n \times \{0\}$ and the orientation-reversing actions corresponding to $\mathcal{E}_n \times Epi(G, C_2)$.*

Proof. By theorem 1.1, every action is equivalent to the action of G by covering transformations on a covering space W of some $N(v)$, and the equivalence classes of actions correspond to the $\mathrm{Aut}(F_n)$-orbits of $\mathcal{G}_n \times \mathbb{V}_n$. Restricting to the $\mathcal{G}_n$-coordinate defines a function $\mathcal{G}_n \times \mathbb{V}_n \to \mathcal{G}_n$ which is $\mathrm{Aut}(F_n)$-equivariant, so there is an induced function on the sets of $\mathrm{Aut}(F_n)$-orbits. Fix an $\mathrm{Aut}(F_n)$-orbit of $\mathcal{G}_n$ and a generating n-vector $(h_1, \ldots h_n)$ that represents it. Each $\mathrm{Aut}(F_n)$-orbit of $\mathcal{G}_n \times \mathbb{V}_n$ that restricts to this element contains a representative of the form $((h_1, \ldots, h_n), (v_1, \ldots, v_n))$. The element $((h_1, \ldots, h_n), (1, \ldots, 1))$ is not equivalent to any other such element, and represents the unique element that corresponds to an orientation-preserving action. By corollary 5.2, $((h_1, \ldots, h_n), (v_1, \ldots, v_n))$ corresponds to an orientation-reversing action if and only if sending h_i to v_i defines a surjective homomorphism from G to C_2. By proposition 5.3, this homomorphism is an invariant of the equivalence class. On the other hand, each element ω of $\mathrm{Epi}(G, C_2)$ determines a choice of v for which $\omega = \omega(v)$, so the equivalence classes of orientation-reversing actions that restrict to the orbit of $(h_1, \ldots, h_n)$ in $\mathcal{G}_n$ correspond to $\mathrm{Epi}(G, C_2)$. □

For classification of orientation-reversing actions up to weak equivalence, there is an added difficulty. An $\mathrm{Aut}(F_n) \times \mathrm{Aut}(G)$-orbit of elements of $\mathcal{G}_n$ is a union of a collection of $\mathrm{Aut}(F_n)$-orbits, say $\{C_1, \ldots, C_r\}$. It produces one weak equivalence class of orientation-preserving actions, but for orientation-reversing actions, one must determine the $\mathrm{Aut}(G)$-orbits of $\{C_1, \ldots, C_r\} \times \mathrm{Epi}(G, C_2)$. This seems to be a subtle problem, in general.

It often happens, however, that $\mathcal{G}_n$ consists of only one $\mathrm{Aut}(F_n)$-orbit, in which case the action of $\mathrm{Aut}(G)$ on $\{C_1\} \times \mathrm{Epi}(G, C_2)$ can be identified with the action on $\mathrm{Epi}(G, C_2)$. Thus in this case, the classification of actions on the orientable handlebody V_g is easy:

Theorem 5.5. *Suppose that all elements of $\mathcal{G}_n$ are Nielsen equivalent, and put $g = 1 + |G|(n-1)$. Then*

(1) *There is only one equivalence class of orientation-preserving free G-actions on V_g.*
(2) *The set of weak equivalence classes of orientation-reversing free actions of G on V_g corresponds bijectively to the set of Aut(G)-orbits of Epi(G, C_2).*

Conjecturally, all generating n-vectors are equivalent whenever G is finite and $n > \mu(G)$ (see the discussion in [13]). So the previous theorem might give a complete classification of all actions on orientable handlebodies above the minimal genus. The conjecture is known for many classes of groups, such as solvable groups [2], $\mathrm{PSL}(2, p)$ (p prime) [4], $\mathrm{PSL}(2, 3^p)$ (p prime) [13], $\mathrm{PSL}(2, 2^m)$ [3], and the Suzuki groups $\mathrm{Sz}(2^{2m-1})$ [3].

A nice example is the quaternion group Q of order 8. One can check that for any $n \geq 2 = \mu(Q)$, any two generating n-vectors of Q are Nielsen equivalent. So for any $k \geq 1$, there is one equivalence class of orientation-preserving free Q-action on V_{1+8k}, and there are three equivalence classes of orientation-reversing free Q-actions, corresponding to the nonzero elements of $\mathrm{Hom}(Q, C_2) = H^1(Q; \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Under the $\mathrm{Aut}(Q)$-action on $\mathrm{Epi}(Q, C_2)$, all three elements lie in the same orbit, so there is only one weak equivalence class of orientation-reversing free Q-action on V_{1+8k}.

Let us finish this section with another example. For $r \geq 3$ let D_r be the dihedral group of $2r$ elements and presentation $\langle s_1, s_2 : s_1^2 = s_2^2 = (s_1 s_2)^r = 1 \rangle$.

Suppose first that $n > 2 = \mu(D_r)$. Since D_r is solvable, there is only one $\mathrm{Aut}(F_n)$-orbit in $\mathcal{G}_n$, and hence there is only one equivalence class of orientation-preserving actions. If r is even, there are three classes of orientation-reversing actions, represented by the elements

$$\{((s_1, s_2, 1, ..., 1), (-1, -1, 1, ..., 1)), ((s_1, s_2, 1, ..., 1), (1, -1, 1, ..., 1)),\\ ((s_1, s_2, 1, ..., 1), (-1, 1, 1, ..., 1))\}$$

of $\mathcal{G}_n \times \mathrm{Epi}(D_r, C_2)$. If r is odd, the second two do not define homomorphisms from D_r to C_2, and there is only one equivalence class. When r is even, there are two weak equivalence classes of orientation-reversing actions, represented by:

$$\{((s_1, s_2, 1, ..., 1), (-1, -1, 1, ..., 1)), ((s_1, s_2, 1, ..., 1), (1, -1, 1, ..., 1))\}.$$

Suppose now that $n = 2$. A set of representatives of the $\mathrm{Aut}(F_2)$-orbits in $\mathcal{G}_2$ is $\{(s_1, (s_1 s_2)^m) : 1 \leq m < r/2, (m, r) = 1\}$ (see theorem 4.5 of [13]). There are $\varphi(r)/2$ classes of orientation-preserving actions (where φ is the Euler function) forming one weak equivalence class. If r is is odd there are $\varphi(r)/2$ classes of orientation-reversing actions. If r is even there are $3\varphi(r)/2$ classes of orientation-reversing actions forming $\varphi(r)$ weak equivalence classes.

6. Actions on nonorientable handlebodies

There is a simple algebraic criterion for G to act freely on the nonorientable handlebody N_m of genus m. Recall that $H^1(G; \mathbb{Z}/2)$ can be identified with $\mathrm{Hom}(G, C_2)$.

Proposition 6.1. *G acts freely on N_m if and only if $m = 1 + |G|(n-1)$ where $n \geq \mu(G)$ and $n > rk\, H^1(G; \mathbb{Z}/2)$.*

Proof. If $n < \mu(G)$ then $\mathcal{G}_n$ is empty and G does not act freely on any handlebody of genus n, so we assume that $n \geq \mu(G)$. According to corollary 5.2, an element $((h_1, \ldots, h_n), (v_1, \ldots, v_n)) \in \mathcal{G}_n \times \mathbb{V}_n$ represents an orbit corresponding to an action on a nonorientable handlebody if and only if sending h_i to v_i does not define a homomorphism from G to C_2. So N_m has no free action exactly when all of the 2^n choices for v define homomorphisms. Since $\mathrm{rk}\, H^1(G; \mathbb{Z}/2) \leq \mu(G) \leq n$, the latter is equivalent to $\mathrm{rk}\, H^1(G; \mathbb{Z}/2) = n$. □

For the quaternion group Q considered in section 5, we have $2 = \mu(Q) = \mathrm{rk}\, H^1(Q; \mathbb{Z})$, so Q acts freely on V_9, but not on N_9.

We may combine proposition 6.1 with theorem 5.4 to determine the genera on which G can act:

Corollary 6.2. *Let $A = \{1 + |G|(n-1) \mid n \geq \mu(G)\}$. Then*

(1) *G acts freely preserving orientation on V_m if and only if $m \in A$.*
(2) *G acts freely reversing orientation on V_m if and only if $m \in A$ and $rk\, H^1(G; \mathbb{Z}/2) > 0$.*
(3) *G acts freely on N_m if and only if $m \in A$ and either $m > 1 + |G|(\mu(G) - 1)$ or $rk\, H^1(G; \mathbb{Z}/2) < \mu(G)$.*

There is a version of theorem 5.5 for actions on nonorientable handlebodies.

Theorem 6.3. *Suppose that $n > \mu(G)$ and that all generating n-vectors of G are Nielsen equivalent. Put $m = 1 + |G|(n-1)$. Then, all free actions of G on N_m are equivalent.*

Proof. Fix a generating set $h_1, \ldots, h_{n-1}$ with $n-1$ elements. Since all generating n-vectors are Nielsen equivalent, each $\mathrm{Aut}(F_n)$-orbit of $\mathcal{G}_n \times \mathbb{V}_n$ has a representative of the form $((h_1, \ldots, h_{n-1}, 1), v)$. Fix such an element corresponding to an action on N_m. Suppose first that $v_n = -1$. For any i with $v_i = 1$, the basic Nielsen move sending h_i to $h_i h_n = h_i$ changes v_i to $v_i v_n = -1$. So $((h_1, \ldots, h_{n-1}, 1), v)$ is equivalent to $((h_1, \ldots, h_{n-1}, 1), (-1, \ldots, -1))$. Suppose that $v_n = 1$. By corollary 5.2, sending each h_i to v_i does not define a homomorphism to C_2, so there is some product $h_{i_1}^{\epsilon_1} \cdots h_{i_k}^{\epsilon_k} = 1$, with all $\epsilon_i = \pm 1$, for which $v_{i_1}^{\epsilon_1} \cdots v_{i_k}^{\epsilon_k} = -1$. A sequence of k basic Nielsen moves replacing h_n by $h_n h_{i_j}^{\epsilon_j}$ shows that $((h_1, \ldots, h_{n-1}, 1), (v_1, \ldots, v_{n-1}, 1))$ is equivalent to $((h_1, \ldots, h_{n-1}, 1), (v_1, \ldots, v_{n-1}, -1))$, which we have seen is equivalent to $((h_1, \ldots, h_{n-1}, 1), (-1, \ldots, -1))$. Therefore we have only one $\mathrm{Aut}(F_n)$-orbit of elements of $\mathcal{G}_n \times \mathbb{V}_n$ that corresponds to an action on a nonorientable handlebody. □

By way of illustration, we return to our example of actions of D_r. If r is odd then $\mathrm{rk}\, H^1(D_r, \mathbb{Z}_2) = 1$ and if r is even then $\mathrm{rk}\, H^1(D_r, \mathbb{Z}_2) = 2$. We have $\mu(D_r) = 2$, and corollary 6.2 shows that D_r acts on N_{2r+1} if and only if r is odd. When r is odd, there are $\varphi(m)/2$ equivalence classes of actions, represented by $((s_1, (s_1 s_2)^m), (-1, -1))$ where m is relatively prime to r and $1 \leq m < r/2$. These form one weak equivalence class. When $n > 2$, there is one equivalence class of actions on $N_{1+2r(n-1)}$ represented by $((s_1, s_2, 1, \ldots, 1), (-1, -1, -1, \ldots, -1))$.

As we noted in section 5, it is conjectured that all generating n-vectors are equivalent whenever G is finite and $n > \mu(G)$, so theorem 6.3 might classify all actions on N_m when $m > 1 + |G|(\mu(G) - 1)$. The classification of actions on the nonorientable handlebody of genus $1 + |G|(\mu(G) - 1)$ seems to be an interesting general problem.

7. Actions of abelian groups

In this section, we will completely classify free actions of abelian groups on handlebodies.

Throughout this section, we assume that G is abelian. For now, write G as $C_{d_1} \oplus \cdots \oplus C_{d_n}$ where $d_{i+1} | d_i$ for $1 \leq i < n$. We have $\mu(G) = n$, since clearly $\mu(G) \leq n$, while $G \otimes C_{d_n} \cong C_{d_n}^n$ requires n generators.

Theorem 4.1 of [13] tells the equivalence classes of generating $\mu(G)$-vectors. Fix a generator s_i for C_{d_i}. Each $\mathrm{Aut}(F_n)$-orbit in $\mathcal{G}_n \times \mathbb{V}_n$ contains exactly one element of the form $(s_1, \ldots, s_{n-1}, s_n^m)$ where m is relatively prime to d_n and $1 \leq m \leq d_n/2$. There is only one weak equivalence class, since for each such m, there is an automorphism of G fixing s_i for $i < n$ and sending s_n to s_n^m.

It will be convenient to rewrite G as $C_{e_1} \oplus \cdots \oplus C_{e_k} \oplus C_{d_1} \oplus \cdots \oplus C_{d_\ell}$, where the e_i are even, the d_j are odd, each $e_{i+1}|e_i$, each $d_{j+1}|d_j$, and $d_1|e_k$. We write s_i for the selected generator of C_{e_i} and t_j for the selected generator of C_{d_j}. There is a corresponding decomposition $\mathbb{V}_n = \mathbb{V}_k \oplus \mathbb{V}_\ell$, in which we will denote elements by $(v, w) = (v_1, \ldots, v_k, w_1, \ldots, w_\ell)$. Also, we write $|\{e_1, \ldots, e_k\}|$ for the cardinality of the set $\{e_1, \ldots, e_k\}$.

We now analyze the $\mathrm{Aut}(F_n)$- and $\mathrm{Aut}(F_n) \times \mathrm{Aut}(G)$-orbits on $\mathcal{G}_n \times \mathbb{V}_n$. Using theorem 4.1 of [13] discussed above, every $\mathrm{Aut}(F_n)$-orbit has a representative of the form $((s_1, \ldots, s_k, t_1, \ldots, t_\ell^m), (v_1, \ldots, v_k, w_1, \ldots, w_\ell))$, or of the form $((s_1, \ldots, s_k^m), (v_1, \ldots, v_k))$ if $\ell = 0$. For such a representative, choose a corresponding free action of G on a handlebody W.

Suppose first that W is orientable. Proposition 5.1 shows that all $w_j = 1$. Each choice of v determines a different homomorphism $\overline{\omega}\colon G \to C_2$, so all the possible choices for v (an element of $\mathbb{V}_k$) and m (an integer relatively prime to d_ℓ with $1 \le m \le d_\ell/2$, or relatively prime to to e_k with $1 \le m \le e_k/2$ if $\ell = 0$) determine inequivalent actions. As in theorem 5.4, the choices with $v = (1, \ldots, 1)$ are the orientation-preserving actions, and all others are orientation-reversing.

Still assuming that W is orientable, we consider weak equivalence. If α is the automorphism of G that sends t_ℓ to t_ℓ^m (or s_k to s_k^m, when $\ell = 0$) then the action of $(1, \alpha)$ sends $((s_1, \ldots, s_k, t_1, \ldots, t_\ell), (v_1, \ldots, v_k, 1, \ldots, 1))$ to $((s_1, \ldots, s_k, t_1, \ldots, t_\ell^m), (v_1, \ldots, v_k, 1, \ldots, 1))$ (or $((s_1, \ldots, s_k), (v_1, \ldots, v_k))$ to $((s_1, \ldots, s_k^m), (v_1, \ldots, v_k)))$, so for weak equivalence we may eliminate the orbit representatives with $m \neq 1$. In particular, there is only one weak equivalence class of orientation-preserving actions. Suppose the action is orientation-reversing, so that some $v_j = -1$. Choose the largest such j.

Suppose that $v_i = 1$ for some e_i for which $e_j|e_i$. Let α be the automorphism of G that sends s_i to $s_i s_j$ and fixes all other generators, and let ρ be the automorphism of F_n that sends x_i to $x_i x_j$ and fixes all other generators. We have $(\rho, \alpha) \cdot ((s_1, \ldots, s_k, t_1, \ldots, t_\ell), (v_1, \ldots, v_i, \ldots, v_j, \ldots v_k, 1, \ldots, 1)) = ((s_1, \ldots, s_k, t_1, \ldots, t_\ell), (v_1, \ldots, v_i v_j, \ldots, v_j, \ldots v_k, 1, \ldots, 1))$. Repeating this for all such i, we may make $v_i = -1$ whenever $e_j|e_i$; that is, after possibly reselecting j to a larger value with the same value of e_j, we may assume that $v_i = -1$ for every $i \le j$, $v_i = 1$ for every $i > j$, and that $e_{j+1} < e_j$ (or $j = k$). Taking only representatives with this property reduces our collection of representatives of $\mathrm{Aut}(F_n) \times \mathrm{Aut}(G)$-orbits to only $|\{e_1, \ldots, e_k\}|$ elements. To check that no two of these can be in the same orbit, we observe that the kernels of the $\overline{\omega}$ for these different elements are not isomorphic. Alternatively we may think in terms of actions: For the action defined by an element in this form, there is a primitive element in $\pi_1(N(v))$ that determines an orientation-reversing covering transformation of W, and whose e_j-th power lifts to an orientation-preserving loop, and e_j is the smallest integer with this property. For every action weakly equivalent to this one, e_j must be the smallest integer with this property.

Suppose now that W is nonorientable, and again consider an orbit representative $((s_1, \ldots, s_k, t_1, \ldots, t_\ell^m), (v_1, \ldots, v_k, w_1, \ldots, w_\ell)) \in \mathcal{G}_n \times \mathbb{V}_n$. Proposition 5.1 shows that some $w_j = -1$. By basic Nielsen moves replacing an s_i (or a t_i) by $s_i t_j$ (or $t_i t_j$) d_j times, we may make every v_i and every w_i equal to -1 (in case $j = \ell$, use t_ℓ^m rather than t_ℓ). Therefore the equivalence classes of actions correspond to the choices for m, and there is only one weak equivalence class.

We now collect these observations.

Theorem 7.1. *Let $G = C_{e_1} \oplus \cdots \oplus C_{e_k} \oplus C_{d_1} \oplus \cdots \oplus C_{d_\ell}$, as above. If $e_k = 2$, put $N = 1$, otherwise put $N = \varphi(e_k)/2$ if $\ell = 0$ and $N = \varphi(d_\ell)/2$ if $\ell > 0$. Then the free actions on handlebodies of minimal genus $1 + |G|(k + \ell - 1)$ are as follows.*

1. *For orientation-preserving actions, there are N equivalence classes, forming one weak equivalence class.*
2. *For orientation-reversing actions, there are $(2^k - 1)N$ equivalence classes, forming $|\{e_1, \ldots, e_k\}|$ weak equivalence classes.*
3. *If $\ell = 0$, then G does not act freely on the nonorientable handlebody. If $\ell > 0$, then there are N equivalence classes, forming one weak equivalence class.*

For actions above the minimal genus, we have:

Theorem 7.2. *For $n > k + \ell$, G acts freely on the orientable and nonorientable handlebodies of genus $1 + |G|(n - 1)$, with the following equivalence classes.*

1. *For orientation-preserving actions, there is one equivalence class.*
2. *For orientation-reversing actions, there are $2^k - 1$ equivalence classes, forming $|\{e_1, \ldots, e_k\}|$ weak equivalence classes.*
3. *For actions on the nonorientable handlebody, there is one equivalence class.*

Proof. Since G is solvable, [2] shows that all generating n-vectors Nielsen are equivalent to $(s_1, \ldots, s_k, t_1, \ldots, t_\ell, 1, \ldots, 1)$. Therefore theorem 5.4 gives part (1) and theorem 6.3 gives (3). For (2), the proof is then essentially the same as that of theorem 7.1; if one allows some of the d_i to equal 1, in effect making $k + \ell = n$, then the proof is almost line-for-line unchanged. □

References

[1] R. Canary and D. McCullough, Homotopy equivalences of 3-manifolds and deformation theory of Kleinian groups, to appear in *Mem. Amer. Math. Soc.*

[2] M. Dunwoody, Nielsen transformations, in *Computational Problems in Abstract Algebra* (Proc. Conf. Oxford, 1967), Pergamon, Oxford (1970), 45–46.

[3] M. Evans, T-systems of certain finite simple groups, *Math. Proc. Cambridge Phil. Soc.* (1993), 9–22.

[4] R. Gilman, Finite quotients of the automorphism group of a free group, *Can. J. Math.* 29 (1977), 541–551.
[5] J. Hempel, 3-*manifolds*, Princeton University Press, 1976.
[6] J. Kalliongis and D. McCullough, Orientation-reversing involutions of handlebodies, *Trans. Amer. Math. Soc.* 348 (1996),1739–1755.
[7] J. Kalliongis and A. Miller, Equivalence and strong equivalence of actions on handlebodies, *Trans. Amer. Math. Soc.* 308 (1988), 721–745.
[8] J. Kalliongis and A. Miller, The symmetries of genus one handlebodies, *Canad. J. Math.* 43 (1991), 371–404.
[9] A. Karrass, A. Pietrowski, and D. Solitar, Finite and infinite cyclic extensions of free groups, *J. Australian Math. Soc.* 16 (1972), 458–466.
[10] D. McCullough, Topological and algebraic automorphisms of 3-manifolds, in *Groups of Self-Equivalences and Related Topics,* ed. R. Piccinini, Springer-Verlag Lecture Notes in Mathematics Vol. 1425 (1990), 102-113.
[11] D. McCullough and A. Miller, Homeomorphisms of 3-manifolds with compressible boundary, *Mem. Amer. Math. Soc.* No. 344 (1986), 1–100.
[12] D. McCullough, A. Miller, and B. Zimmermann, Group actions on handlebodies, *Proc. London Math. Soc.* (3) 59 (1989), 373–416.
[13] D. McCullough and M. Wanderley, Free actions on handlebodies, *J. Pure Appl. Algebra.* 181 (2003), 85–104.
[14] A. Miller and B. Zimmermann, Large groups of symmetries of handlebodies, *Proc. Amer. Math. Soc.* 106 (1989), 829–838.
[15] J. Nielsen, Die Isomorphismengruppe der freien gruppe, *Math. Ann.* 91 (1924), 169–209.
[16] J. H. Przytycki, Free actions of $\mathbb{Z}_n$ on handlebodies, *Bull. Acad. Polonaise des Sciences* XXVI (1978), 617–624.
[17] B. Zimmermann, Finite maximal orientation reversing group actions on handlebodies and 3-manifolds, *Rend. Circ. Mat. Palermo* (2) 48 (1999), 549–562.

Departamento de Matematicas Fundamentales, Facultad de Ciencias, Universidad Nacional de Educación a Distancia, Madrid 28040, SPAIN
E-mail address: acosta@mat.uned.es

Department of Mathematics, University of Oklahoma, Norman, Oklahoma 73019, USA
E-mail address: dmccullough@math.ou.edu
URL: www.math.ou.edu/~dmccullough/