Fiber-preserving imbeddings and
diffeomorphisms

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1 Introduction

Let $\text{Diff}(M)$ be the group of diffeomorphisms of a smooth manifold $M$, with the $C^\infty$-topology. For a smooth submanifold $N$ of $M$, denote by $\text{Imb}(N,M)$ the space of all smooth imbeddings $j$ of $N$ into $M$ such that $j^{-1}(\partial M) = N \cap \partial M$. In [10], R. Palais proved a useful result relating diffeomorphisms and imbeddings. In the case when $M$ is closed, it says that if $W \subset V$ are submanifolds of $M$, then the mappings $\text{Diff}(M) \to \text{Imb}(V,M)$ and $\text{Imb}(V,M) \to \text{Imb}(W,M)$ obtained by restricting diffeomorphisms and imbeddings are locally trivial, and hence are Serre fibrations. The same results, with variants for manifolds with boundary and more complicated additional boundary structure, were proven by J. Cerf in [1]. Among various applications of these results, the Isotopy Extension Theorem follows by lifting a path in $\text{Imb}(V,M)$ starting at the inclusion map of $V$ to a path in $\text{Diff}(M)$ starting at $1_M$. Moreover, parameterized versions of isotopy extension follow just as easily from the homotopy lifting property for $\text{Diff}(M) \to \text{Imb}(V,M)$ (see corollary 5.3).

In the common situation of a fibering of manifolds, it is natural to consider the spaces of imbeddings and diffeomorphisms that respect the fibered structure. Consider a (smooth) fibering $p: E \to B$ of compact manifolds, possibly with boundary. (Actually, most of our results allow $E$ and $B$ to be non-compact, although the fiber and the relevant submanifolds must be assumed to be compact. Also, we prove versions with control relative to subsets of the boundary of $B$ and their preimages in $E$. For clarity we omit such complications in this introductory discussion.) A diffeomorphism of $E$ is called fiber-preserving when it takes each fiber of $E$ to a fiber of $E$, and vertical when it takes each fiber to itself. The space $\text{Diff}_f(E)$ of fiber-preserving diffeomorphisms of $E$ contains the subspace $\text{Diff}_v(E)$ of vertical diffeomorphisms. Any fiber-preserving diffeomorphism $g$ of $E$ induces a diffeomorphism $\tilde{g}$ of $B$, and this defines a map from $\text{Diff}_f(E)$ to $\text{Diff}(B)$ for which the preimage of the identity map is $\text{Diff}_v(E)$. In section 5 we prove

**Projection Theorem** (Theorem 5.2) $\text{Diff}_f(E) \to \text{Diff}(B)$ is locally trivial.

This theorem is essentially due to W. Neumann and F. Raymond (see the comments below). The homotopy extension property for the projection fibration translates directly into the following.

**Parameterized Isotopy Extension Theorem** (Corollary 5.3) Suppose that
§1. Introduction

$p: E \to B$ is a fibering of compact manifolds, and suppose that for each $t$ in a path-connected parameter space $P$, there is an isotopy $g_{t,s}$ such that $g_{t,0}$ lifts to a diffeomorphism $G_{t,0}$ of $E$. Assume that sending $(t,s) \to g_{t,s}$ defines a continuous function from $P \times [0,1]$ to $\text{Diff}(B)$ and sending $t$ to $G_{t,0}$ defines a continuous function from $P$ to $\text{Diff}(E)$. Then the family $G_{t,0}$ extends to a continuous family on $P \times I$ such that for each $(t,s)$, $G_{t,s}$ is a fiber-preserving diffeomorphism inducing $g_{t,s}$ on $B$.

A submanifold of $E$ is called vertical if it is a union of fibers, and in this case it will be assumed to have the fibered structure so that the inclusion map is fiber-preserving. An imbedding of a fibered manifold $W$ into $E$ is called fiber-preserving when the image of each fiber of $W$ is a fiber of $E$. The space of all fiber-preserving imbeddings from $W$ to $E$ is denoted by $\text{Imb}_f(W,E)$. When $W \subseteq E$, $\text{Imb}_f(W,E)$ contains the subspace of vertical imbeddings $\text{Imb}_v(W,E)$ which take each fiber to itself. For fiber-preserving and vertical imbeddings of vertical submanifolds, we have a more direct analogue of Palais’ results.

**Restriction Theorem** (Corollary 6.5) Let $V$ and $W$ be vertical submanifolds of $E$ with $W \subseteq V$, each of which is either properly imbedded or codimension-zero. Then the restrictions $\text{Imb}_f(V,E) \to \text{Imb}_f(W,E)$ and $\text{Imb}_v(V,E) \to \text{Imb}_v(W,E)$ are locally trivial.

As shown in theorem 6.6, the Projection and Restriction Theorems can be combined into a single commutative square in which all four maps are locally trivial:

$$
\begin{array}{ccc}
\text{Diff}(E) & \longrightarrow & \text{Imb}_f(W,E) \\
\downarrow & & \downarrow \\
\text{Diff}(B) & \longrightarrow & \text{Imb}(p(W),B) .
\end{array}
$$

In 3-dimensional topology, a key role is played by manifolds admitting a more general kind of fibered structure, called a Seifert fibering. Some general references for Seifert-fibered 3-manifolds are [3, 4, 5, 7, 8, 13, 14, 15, 16]. In section 8, we prove the analogues of the results discussed above for many Seifert fiberings $p: \Sigma \to \mathcal{O}$, not necessarily 3-dimensional. Actually, we work in a somewhat more general context, called singular fiberings, which resemble Seifert fiberings but for which none of the usual structure of the fiber as a homogeneous space is required.

In the late 1970’s fibration results akin to our Projection Theorem for the singular fibered case were proven by W. Neumann and F. Raymond [6]. They
were interested in the case when \( \Sigma \) admits an action of the \( k \)-torus \( T^k \) and \( \Sigma \to \mathcal{O} \) is the quotient map to the orbit space of the action. They proved that the space of (weakly) \( T^k \)-equivariant homeomorphisms of \( \Sigma \) fibers over the space of homeomorphisms of \( \mathcal{O} \) that respect the orbit types associated to the points of \( \mathcal{O} \). A detailed proof of this result when the dimension of \( \Sigma \) is \( k + 2 \) appears in the dissertation of C. Park [11]. Park also proves analogous results for space of weakly \( G \)-equivariant maps for principal \( G \)-bundles and for Seifert fiberings of arbitrary dimension [11, 12]. These results do not directly overlap ours since we always consider the full group of fiber-preserving diffeomorphisms without any restriction to \( G \)-equivariant maps (indeed, no assumption of a \( G \)-action is even present).

Some technical applications of our results appear in [9]. In the present paper we give one main application. For a Seifert-fibered manifold \( \Sigma \), \( \text{Diff}(\Sigma) \) acts on the set of Seifert fiberings, and the stabilizer of a given fibering is \( \text{Diff}_f(\Sigma) \), thus the space of cosets \( \text{Diff}(\Sigma)/\text{Diff}_f(\Sigma) \) is the space of Seifert fiberings of \( \Sigma \). We prove in section 10 that for a Seifert-fibered Haken 3-manifold, each component of the space of Seifert fiberings is weakly contractible (apart from a small list of well-known exceptions, the space of Seifert fiberings is connected). This result is originally due to Neumann and Raymond, since it is an immediate consequence of the results in [6] combined with contemporaneous work of Hatcher [2]. We make the same use of [2].

Our results will be proven by adapting the method developed in [10]. The main new idea needed for the fibered case is a modification of the usual exponential map, called the aligned exponential map \( \text{Exp}_a \). This is defined and discussed in section 4. Section 2 contains some preliminaries needed for carrying out Palais’ approach for manifolds with boundary. In section 3, we reprove the main result of [10] for manifolds which may have boundary. This duplicates [1] (in fact, the boundary control there is more refined than ours), but is included to furnish lemmas as well as to exhibit a prototype for the approach we use to deal with the bounded case in our later settings. In section 7, we give the analogues of the results of Palais and Cerf for smooth orbifolds, which for us are quotients \( \tilde{\mathcal{O}}/H \) where \( \tilde{\mathcal{O}} \) is a manifold and \( H \) is a group acting smoothly and properly discontinuously on \( \tilde{\mathcal{O}} \). Besides being of independent interest, these analogues are needed for the case of singular fiberings.

By a submanifold \( N \) of \( M \), we mean a smooth submanifold. When \( M \) has boundary and \( \dim(N) < \dim(M) \), we always require that \( N \) be properly imbedded in the sense that \( N \cap \partial M = \partial N \). If \( N \) has codimension 0, we require
that the frontier of $N$ be a codimension-1 submanifold of $M$. In particular, it is understood that the elements of $\text{Imb}(N, M)$ carry $N$ to a submanifold satisfying these conditions. The notation $\text{Diff}(M \text{ rel } \partial M)$ means the space of diffeomorphisms which restrict to the identity map on each point of $\partial M$, and for $X \subseteq M$, $\text{Imb}(X, M \text{ rel } \partial M)$ means the imbeddings that equal the inclusion $X \cap \partial M$. For $K \subseteq M$, $\text{Diff}^K(M)$ means the diffeomorphisms that agree with the identity on $M - K$. We say that $K$ is a neighborhood of the subset $X$ when $X$ is contained in the topological interior of $K$. If $K$ is a neighborhood of a submanifold $N$, then $\text{Imb}^K(N, M)$ means the elements $j$ in $\text{Imb}(N, M)$ such that $K$ is a neighborhood of $j(N)$.

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2 Metrics which are products near the boundary

When $M$ has a Riemannian metric, we denote by $d$ the associated topological metric defined by putting $d(x, y)$ equal to the infimum of the lengths of all piecewise differentiable paths from $x$ to $y$ when $x$ and $y$ lie in the same component of $M$, and equal to 1 if $x$ and $y$ lie in different components.

Let $V$ be a (possibly empty) compact submanifold of $M$. Recall that we always assume that $V$ is properly imbedded, if it has positive codimension, or that the frontier of $V$ is a properly imbedded codimension-1 submanifold, if $V$ has codimension 0. Fix a smooth collar $\partial M \times [0, 2]$ of $\partial M$ such that $V \cap \partial M \times [0, 2]$ is a union of $[0, 2]$-fibers. Such a collar can be obtained by constructing an inward-pointing vector field on a neighborhood of $\partial M$ which is tangent to $V$, and using the integral curves associated to the vector field to produce the collar. On $\partial M \times [0, 2]$, fix a Riemannian metric that is the product of a metric on $\partial M$ and the usual metric on $[0, 2)$. Form a metric on $M$ from this metric and any metric defined on all of $M$ using a partition of unity subordinate to the open cover $\{\partial M \times [0, 2], M - \partial M \times I\}$, where $I = [0, 1]$. Such a metric is said to be a product near $\partial M$ such that $V$ meets the collar $\partial M \times I$ in $I$-fibers. It has the following properties for $0 \leq t \leq 1$:

(i) If $x \in M - \partial M \times I$, then $d(x, \partial M) > 1$.

(ii) If $x = (y, t) \in \partial M \times I$, then $d(x, \partial M) = t$. 

(iii) If $V$ has positive codimension, then for any tubular neighborhood of $V$ obtained by exponentiating a normal bundle of $V$, the fiber at each point of $V \cap \partial M \times \{t\}$ lies in $\partial M \times \{t\}$. In particular, the fiber of each point in $V \cap \partial M$ lies in $\partial M$. If $V$ has codimension 0, then the corresponding statement holds for any tubular neighborhood of the frontier of $V$.

A Riemannian metric is called **complete** if every Cauchy sequence converges. For a complete Riemannian metric on $M$, a geodesic can be extended indefinitely unless it reaches a point in the boundary of $M$, where it may continue or it may fail to be extendible because it “runs out of the manifold.”

One may obtain a complete metric on $M$ that is a product near $\partial M$ such that $V$ meets the collar $\partial M \times I$ in $I$-fibers as follows. Carry out the previous construction using a metric on $\partial M \times [0,2)$ that is the product of a complete metric on $\partial M$ and the standard metric on $[0,2)$. Define $f: M - \partial M \to (0,\infty)$ by putting $f(x)$ equal to the supremum of the values of $r$ such that $\operatorname{Exp}$ is defined on all vectors in $T_x(M)$ of length less than $r$. Let $g: M - \partial M \to (0,\infty)$ be a smooth map that is an $\epsilon$-approximation to $1/f$, and let $\phi: M \to [0,1]$ be a smooth map which is equal to 0 on $\partial M \times I$ and is 1 on $M - \partial M \times [0,2)$. Give $M \times [0,\infty)$ the product metric, and define a smooth imbedding $i: M \to M \times [0,\infty)$ by $i(x) = (x, \phi(x)g(x))$ if $x \notin \partial M$ and $i(x) = (x,0)$ if $x \in \partial M$. The restricted metric on $i(M)$ agrees with the original metric on $\partial M \times I$ and is complete.

From now on, all metrics will be assumed to be complete.

### 3 The Palais-Cerf restriction theorem

In this section we modify some results from [10] to apply to the bounded case. This duplicates [1], in fact our results are not as general since we do not work in the setting of manifolds with corners. On the other hand, our argument will provide lemmas needed for the fibered cases, and is the prototype for the approach we use to deal with the bounded case in our later settings.

Let $X$ be a $G$-space and $x_0 \in X$. A **local cross-section** for $X$ at $x_0$ is a map $\chi$ from a neighborhood $U$ of $x_0$ into $G$ such that $\chi(u)x_0 = u$ for all $u \in U$. By replacing $\chi(u)$ by $\chi(u)\chi(x_0)^{-1}$, one may always assume that $\chi(x_0) = 1_G$. If $X$ admits a local cross-section at each point, it is said to admit local cross-sections. From [10] we have
Proposition 3.1 Let $G$ be a topological group and $X$ a $G$-space admitting local cross-sections. Then any equivariant map of a $G$-space into $X$ is locally trivial.

In fact, when $\pi: Y \to X$ is $G$-equivariant, the local coordinates on $\pi^{-1}(U)$ are just given by sending the point $(u, z) \in U \times \pi^{-1}(y_0)$ to $\chi(u) \cdot z$. Some additional properties of the bundles obtained in proposition 3.1 are given in [10].

The following technical lemma will simplify some of our applications of proposition 3.1.

Proposition 3.2 Let $V$ be a submanifold of $M$, let $I(V, M)$ be a space of imbeddings of $V$ into $M$, and let $D(M)$ a group of diffeomorphisms of $M$. Suppose that for every $i \in I(V, M)$, the space of imbeddings $I(i(V), M)$ has a local $D(M)$ cross-section at the inclusion map of $i(V)$ into $M$. Then $I(V, M)$ has local cross-sections.

Proof of 3.2: Denote by $j_{i(V)}$ the inclusion map of $i(V)$ into $M$. Let $i \in I(V, M)$ and define $Y: I(V, M) \to I(i(V), M)$ by $Y(j) = ji^{-1}$. For a local cross-section $\chi: U \to D(M)$ at $j_{i(V)}$, define $Y_1$ to be the restriction of $Y$ to $Y^{-1}(U)$, a neighborhood of $i$ in $I(V, M)$. Then $\chi Y_1: Y^{-1}(U) \to D(M)$ is a local cross-section for $I(V, M)$ at $i$. For if $j \in Y^{-1}(U)$ and $x \in V$, then $\chi(Y_1(j))(i(x)) = \chi(Y_1(j))(j_{i(V)}(i(x))) = Y_1(j)(i(x)) = j(x)$.

The results in [10] depend in large part on three lemmas, called lemmas b, c and d there. Here, we adapt their statements and proofs to the context of manifolds with boundary. First, for $L \subseteq M$ define $\text{Maps}^L(M, M)$ be the space of smooth maps $f: M \to M$ such that $f(\partial M) \subseteq \partial M$ and $f(x) = x$ for all $x \in M - L$.

Lemma 3.3 (Palais’ lemma b) Let $K$ be a compact subset of a manifold $M$. Then there exists a neighborhood $J$ of $1_M$ in $\text{Maps}^K(M, M)$ which consists of diffeomorphisms.

Proof of 3.3: There exists a ($C^\infty$-) neighborhood $N$ of the identity consisting of maps $f$ for which the differential $T_x(f)$ is an isomorphism for all $x$. Since $f(\partial M) \subseteq \partial M$, this implies that $f$ is a local diffeomorphism. Since $K$
is compact, the preimage of any compact subset of $M$ under $f$ is compact. Therefore $f$ is a covering map. If $M \neq K$, then this covering must be 1-fold off of $f(K)$, hence must be a diffeomorphism, so assume that $M$ is compact. Fix $\epsilon > 0$ such that no closed noncontractible loop in $M$ has length less than $4\epsilon$, and let $J$ consist of the elements in $N$ such that $d(f(x), x) < \epsilon$ for all $x$. Suppose for contradiction that $f \in J$ but $f(p) = f(q)$ for $p \neq q$. Then $d(p, q) < 2\epsilon$, and if $\alpha$ is a geodesic from $p$ to $q$ of length less than $2\epsilon$, then the diameter of $f(\alpha)$ is less than $4\epsilon$. Therefore $f(\alpha)$ is a contractible loop, a contradiction.

3.3

For the next lemmas, we set some notation. The projection from the tangent bundle $T(M)$ to $M$ is denoted by $\pi$. For a submanifold $V$ of $M$, let $X(V, T(M))$ denote the sections $X$ from $V$ to $T(M)|_V$ such that

1. if $x \in V \cap \partial M$, then $X(x)$ is tangent to $\partial M$, and
2. $\text{Exp}(X(x))$ is defined for all $x \in V$.

When the metric is a product near the boundary, property (1) implies that if $x \in V \cap \partial M$, then $\text{Exp}(X(x)) \in \partial M$.

A zero section will usually be denoted by $Z$. The vector fields satisfying (1) and (2) (i.e. the case $V = M$) are denoted simply by $X(T(M))$. When $L$ is a subset of $M$, denote by $X^L(T(M))$ the elements of $X(T(M))$ which agree with $Z$ outside of $L$. A subscript "< $\delta$" indicates the sections such that each image vector has length less than the positive number $\delta$, thus for example

$$X_{<1/2}(V, T(M)) = \{ X \in X(V, T(M)) \mid \|X(x)\| < 1/2 \text{ for all } x \in V \}.$$

**Lemma 3.4** (Palais’ lemma c) Let $V$ be a compact submanifold of the smooth manifold $M$ and $L$ a neighborhood of $V$ in $M$. Assume that the metric on $M$ is a product near $\partial M$ such that $V$ is vertical in $\partial M \times I$. Then there exists a continuous map $k: X_{<1/2}(V, T(M)) \to X^L(T(M))$ such that $k(X)(x) = X(x)$ for all $x$ in $V$ and all $X \in X_{<1/2}(V, T(M))$. Moreover, $k(Z) = Z$, and if $S \subseteq \partial M$ is a closed neighborhood in $\partial M$ of $S \cap \partial V$, and $X(x) = Z(x)$ for all $x \in S \cap \partial V$, then $k(X)(x) = Z(x)$ for all $x \in S$. 


Proof of 3.4: Suppose first that $V$ has positive codimension. Let $\nu_\epsilon(V)$ denote the vectors of length less than $\epsilon$ in the normal bundle of $V$. Fix $\epsilon < 1/2$ and sufficiently small so that $j: \nu_\epsilon(V) \to M$ defined by exponentiation is a tubular neighborhood of $V$ contained in $L$, and such that the union of the fibers at points in $S \cap \partial V$ is contained in $S$, and the union of the fibers at points in $(\partial M - S) \cap \partial V$ is contained in $\partial M - S$. By property (iii) of the metric, the fiber of this neighborhood at each point of $V \cap \partial M \times \{t\}$ lies in $\partial M \times \{t\}$. Since $V$ is compact, we may choose $\epsilon$ sufficiently small so that $j(\omega) \in \partial M \times I$ only when $\pi(\omega) \in \partial M \times I$.

Suppose $v \in T_x(M)$ and that $\text{Exp}(v)$ is defined. For all $u \in T_x(M)$ define $P(u, v)$ to be the vector that results from parallel translation of $u$ along the path that sends $t$ to $\text{Exp}(tv)$, $0 \leq t \leq 1$. Note that $P(u, Z(x)) = u$ for all $u$. Let $\alpha: M \to [0, 1]$ be a smooth function which is identically 1 on $V$ and identically 0 on $M - j(\nu_{\epsilon/2}(V))$. Define $k: X_{<1/2}(V, T(M)) \to X^{L(T(M))}$ by

$$k(X)(x) = \begin{cases} \alpha(x)P(X(\pi(j^{-1}(x))), j^{-1}(x)) & \text{for } x \in j(\nu_\epsilon(V)) \\ Z(x) & \text{for } x \in M - j(\nu_{\epsilon/2}(V)). \end{cases}$$

For $x \in V$, $j^{-1}(x) = Z(x)$ and $\alpha(x) = 1$, so $k(X)(x) = X(x)$. We must check that $k(X) \in X^{L(T(M))}$. Since the metric on $M$ is a product near $\partial M$, $k(X)$ satisfies condition (1) to be in $X(T(M))$. To verify that it satisfies condition (2), fix $x$ such that $k(X)(x) \neq Z(x)$. Suppose first that $x = (y, t) \in \partial M \times I$. Then $\pi(j^{-1}(x))$ has the form $(y', t)$. If $t \geq 1/2$ then $\|P(X(\pi(j^{-1}(x))), j^{-1}(x))\| = \|X(\pi(j^{-1}(x)))\| < 1/2$, so since $\alpha(x) \leq 1$, $\|k(X)(x)\| \leq t = d(x, \partial M)$ and $\text{Exp}(k(X)(x))$ is defined. Suppose $t \leq 1/2$. Since the metric is a product near $\partial M$, the component of $k(X)(x)$ in the $I$-direction can be identified with the component of $X(\pi(j^{-1}(x)))$ in the $I$-direction, so $\text{Exp}(k(X)(x))$ is defined when $\text{Exp}(X(\pi(j^{-1}(x))))$ is. Finally, if $x \notin \partial M \times I$ then also $\pi(j^{-1}(x)) \notin \partial M \times I$ so $d(\pi(j^{-1}(x)), \partial M) > 1$. Since $\|X(\pi(j^{-1}(x)))\| < 1/2$ and $\|j^{-1}(x)\| < \epsilon < 1/2$, $\text{Exp}(k(X)(x))$ is a point that lies within distance 1 of $\pi(j^{-1}(x))$. The fact that $k(Z) = Z$ is immediate from the definition. Finally, if $X(x) = Z(x)$ for all $x \in S \cap \partial V$, then since the metric is a product near the boundary it follows that $k(X)(x) = Z(x)$ for all $x \in S$.

Now suppose that $V$ has codimension zero, so that its frontier $W$ is a properly imbedded submanifold. Let $\nu^I_\epsilon(W)$ denote the vectors of lengths less than $\epsilon$ in the normal bundle of $W$ that exponentiate into $\overline{M - V}$. Proceed as
before, but define $k$ by

$$k(X)(x) = \begin{cases} X(x) & \text{for } x \in V \\ \alpha(x)P(X(\pi(j^{-1}(x))), j^{-1}(x)) & \text{for } x \in j(\nu_i^+(V)) \\ Z(x) & \text{for } x \in M - j(\nu_{i/2}^+(V)) \end{cases}.$$  

For our proof of lemma d, we introduce some additional notation. Assume that the metric on $M$ is selected to be a product near $\partial M$. For $x \notin \partial M \times I$, let $R(x, \epsilon)$ be the set of vectors in $T_x(M)$ of length less than $\epsilon$. If $x = (y, t) \in \partial M \times I$, give $T_x(M)$ coordinates $\omega_1, \ldots, \omega_n$ so that $\omega_1, \ldots, \omega_{n-1}$ are in the $\partial M$-direction (and hence exponentiate into $\partial M \times \{t\}$), and $\omega_n$ is the coordinate in the $I$-direction. Then, define $R(x, \epsilon)$ to be $\{\omega = (\omega_1, \ldots, \omega_n) \in T_x(M) | \|\omega\| < \epsilon$ and $\omega_n \geq -t\}$. For small $\epsilon$ the exponential map $\text{Exp}$ carries $R(x, \epsilon)$ diffeomorphically to an open neighborhood of $x$, even when $x \in \partial M$.

For a properly imbedded submanifold $V$ of $M$ which meets $\partial M \times I$ in $I$-fibers, define $N_\epsilon(V) \subset T(M)|_V$ by $N_\epsilon(V) = \cup_{x \in V} R(x, \epsilon)$. When $V$ is compact, there exists a positive $\epsilon$ such that for every $x \in V$, $\text{Exp}$ carries each $N_\epsilon(V) \cap T_x(M)$ diffeomorphically to a neighborhood of $x \in M$.

For spaces of imbeddings, a “$< \delta$” subscript indicates the imbeddings that are $\delta$-close to the inclusion, thus for example

$$\text{Imb}_{<\delta}(V, M) = \{j \in \text{Imb}(V, M) | d(j(x), i_V(x)) < \delta \text{ for all } x \in V\}.$$  

**Lemma 3.5** (Palais’ lemma d) Assume that the metric on $M$ is a product near $\partial M$. Let $V$ be a compact submanifold of $M$ that meets $\partial M \times I$ in $I$-fibers. For sufficiently small positive $\delta$, there exists a continuous map $X: \text{Imb}_{<\delta}(V, M) \to \mathcal{X}_{<1/2}(V, T(M))$ such that $\text{Exp}(X(j))(x) = j(x)$ for all $x \in V$ and $j \in \text{Imb}_{<\delta}(V, M)$. Moreover, if $j(x) = i_V(x)$ then $X(j)(x) = Z(x)$.

**Proof of 3.5:** Choose $\epsilon < 1/2$ small enough so that for all $x \in V$, $\text{Exp}$ gives a diffeomorphism from $N_\epsilon(V) \cap T_x(M)$ to a neighborhood of $x$ in $M$. Choose $\delta$ small enough so that if $j \in \text{Imb}_{<\delta}(V, M)$ then $j(x) \in \text{Exp}(N_\epsilon(V) \cap T_x(M))$. For $j \in \text{Imb}_{<\delta}(V, M)$ define $X(j)(x)$ to be the unique vector in $N_\epsilon(V) \cap T_x(M)$ such that $\text{Exp}(X(j)(x))$ equals $j(x)$. We must verify that $X \in \mathcal{X}_{<1/2}(V, T(M))$.

Property (1) holds because the metric is a product near $\partial M$, so for $x \in \partial M$ and short vectors $\omega \in T_x(M)$, $\text{Exp}(\omega) \in \partial M$ if and only if $\omega$ is tangent to $\partial M$. Property (2) and the final sentence of the lemma are immediate.
Before giving the main results of this section, we fix some notation to simplify their statements. Suppose $V$ is a compact submanifold of $M$, and $S \subseteq \partial M$ is a (possibly empty) closed subset which is a neighborhood in $\partial M$ of $S \cap \partial V$. Note that this implies that $S \cap \partial V$ is a union of components of $V \cap \partial M$.

In this situation, $\text{Imb}(V, M_{\text{rel}} S)$ will stand for the space of imbeddings that equal the inclusion on $V \cap S$ and carry $V \cap (\partial M - S)$ into $\partial M - S$. As usual, $\text{Imb}^L(V, M_{\text{rel}} S)$ denotes the subspace consisting of all $j$ such that $j(V)$ lies in the topological interior of $L$.

The fundamental result is the analogue of theorem B of [10]. For its proof we make one more definition. Define $F: \mathcal{X}(T(M)) \to \text{Maps}^L(M, M)$ by $F(X)(x) = \text{Exp}(X(x))$. We recall that condition (1) of the definition of $\mathcal{X}(T(M))$ and the fact that the metric is a product near the boundary guarantee that $F(X)(\partial M) \subset \partial M$. Since $\text{Exp}$ is smooth, it follows as in lemma a of [10] that $F$ is continuous.

**Theorem 3.6** Let $V$ be a compact submanifold of $M$, and let $S \subseteq \partial M$ be a closed neighborhood in $\partial M$ of $S \cap \partial V$. Let $L$ be a neighborhood of $V$ in $M$. Then $\text{Imb}^L(V, M_{\text{rel}} S)$ admits local $\text{Diff}^L(M_{\text{rel}} S)$ cross-sections.

**Proof of 3.6:** By proposition 3.2 it suffices to find a local cross-section at the inclusion map $i_V$. Fix a compact neighborhood $K$ of $V$ with $K \subseteq L$. Using lemmas 3.5 and 3.4, we obtain $X: \text{Imb}_{<\delta}(V, M) \to \mathcal{X}_{<1/2}(V, T(M))$ and $k: \mathcal{X}_{<1/2}(V, T(M)) \to \mathcal{X}(T(M))$. Let $J$ be a neighborhood of $1_M$ in $\text{Maps}^K(M, M)$ as in lemma 3.3, and define $U = (FkX)^{-1}(J)$. Then $\chi = FkX: U \to \text{Diff}(M)$ is the desired local $\text{Diff}^L(M_{\text{rel}} S)$ cross-section at $i_V$.

From proposition 3.1 we have immediate corollaries.

**Corollary 3.7** Let $V$ be a compact submanifold of $M$. Let $S \subseteq \partial M$ be a closed neighborhood in $\partial M$ of $S \cap \partial V$, and $L$ a neighborhood of $V$ in $M$. Then the restriction $\text{Diff}^L(M_{\text{rel}} S) \to \text{Imb}^L(V, M_{\text{rel}} S)$ is locally trivial.

**Corollary 3.8** Let $V$ and $W$ be compact submanifolds of $M$, with $W \subseteq V$. Let $S \subseteq \partial M$ a closed neighborhood in $\partial M$ of $S \cap \partial V$, and $L$ a neighborhood
of \( V \) in \( M \). Then the restriction \( \text{Imb}^k(V, M \text{ rel } S) \to \text{Imb}^k(W, M \text{ rel } S) \) is locally trivial.

4 The vertical and aligned exponentials

Let \( p: E \to B \) be a locally trivial smooth map of manifolds, with compact fiber, and let \( \pi: T(E) \to E \) denote the tangent bundle of \( E \). At each point \( x \in E \), let \( V_x(E) \) denote the vertical subspace of \( T_x(E) \) consisting of vectors tangent to the fiber of \( p \). When \( E \) has a Riemannian metric, the orthogonal complement \( H_x(E) \) of \( V_x(E) \) in \( T_x(E) \) is called the horizontal subspace. We usually abbreviate \( V_x(E) \) and \( H_x(E) \) to \( V_x \) and \( H_x \), and call their elements vertical and horizontal respectively. Clearly \( V_x \) is the kernel of \( p_\ast: T_x(E) \to T_{p(x)}(B) \), while \( p_\ast|_{H_x}: H_x \to T_{p(x)}(B) \) is an isomorphism. Each vector \( \omega \in T_x(E) \) has an orthogonal decomposition \( \omega = \omega_v + \omega_h \).

Define the horizontal boundary \( \partial_h E \) to be \( \bigcup_{x \in B} \partial(p^{-1}(x)) \), and the vertical boundary \( \partial_v E \) to be \( p^{-1}(\partial B) \).

A path \( \alpha \) in \( E \) is called horizontal if \( \alpha'(t) \in H_{\alpha(t)} \) for all \( t \) in the domain of \( \alpha \). Let \( \gamma: [a, b] \to B \) be a path such that \( \gamma'(t) \) never vanishes, and let \( x \in E \) with \( p(x) = \gamma(a) \). A horizontal path \( \tilde{\gamma}: [a, b] \to E \) such that \( \tilde{\gamma}(a) = x \) and \( p\tilde{\gamma} = \gamma \) is called a horizontal lift of \( \gamma \) starting at \( x \).

To ensure that horizontal lifts exist, we will need a special metric on \( E \). Using the local product structure, at each point \( x \) in \( \partial_h E \) select a vector field defined on a neighborhood of \( x \) that

(a) points into the fiber at points of \( \partial_h E \), and

(b) is tangent to the fibers wherever it is defined.

Note that by (b), the vector field must be tangent to \( \partial_v E \) at points in \( \partial_h E \). Since scalar multiples and linear combinations of vectors satisfying these two conditions also satisfy them, we may piece these local fields together using a partition of unity to construct a vector field, nonvanishing on a neighborhood of \( \partial_h E \), that satisfies (a) and (b). Using the integral curves associated to this vector field we obtain a smooth collar neighborhood \( \partial_h E \times [0, 2] \) of \( \partial_h E \) such that each \( [0, 2] \)-fiber lies in a fiber of \( p \). On \( \partial_h E \times [0, 2] \), fix a Riemannian metric that is the product of a metric on \( \partial_h E \) and the usual metric on \( [0, 2] \). Form a metric on \( E \) from this metric and any metric on all of \( E \) using a
partition of unity subordinate to the open cover \( \{ \partial_h E \times [0, 2], E - \partial_h E \times I \} \).
Such a metric is said to be a **product near \( \partial_h E \) such that the \( I \)-fibers of \( \partial_h E \times I \) are vertical.** It has the following properties for \( 0 \leq t \leq 1 \):

(i) If \( x \in E - \partial_h E \times I \), then \( d(x, \partial_h E) > 1 \).

(ii) If \( x = (y, t) \in \partial_h E \times \{t\} \), then \( d(x, \partial_h E) = t \).

(iii) For \( x \in \partial_h E \times \{t\} \), the horizontal subspace \( H_x \) is tangent to \( \partial_h E \times \{t\} \).

To see property (iii), start with the fact that \( H_x \) is perpendicular to the fiber \( p^{-1}(p(x)) \). Since the \( I \)-fiber of \( \partial_h E \times I \) that contains \( x \) lies in \( p^{-1}(p(x)) \), \( H_x \) is orthogonal to that \( I \)-fiber as well. Since \( \partial_h E \times \{t\} \) meets the \( I \)-fiber orthogonally, with codimension 1, \( H_x \) is tangent to \( \partial_h E \times \{t\} \).

Property (iii) implies that a horizontal lift starting in some \( \partial_h E \times \{t\} \) will continue in \( \partial_h E \times \{t\} \). Using the compactness of the fiber, the existence of horizontal lifts will then be guaranteed.

**Lemma 4.1** Assume that the metric on \( E \) is a product near \( \partial_h E \) such that the \( I \)-fibers of \( \partial_h E \times I \) are vertical. Let \( \gamma: [a, b] \rightarrow B \) be a path such that \( \gamma'(t) \) never vanishes, and let \( x \in E \) with \( p(x) = \gamma(a) \). Then there exists a unique horizontal lift of \( \gamma \) starting at \( x \).

**Proof of 4.1:** For any horizontal lift \( \tilde{\gamma}(t) \), each \( \tilde{\gamma}'(t) \) is uniquely determined, so the lift through a given point in \( E \) is unique if it exists. For each \( \gamma(t) \), let \( F_{\gamma(t)} \) be the fiber over \( \gamma(t) \). From the local theory of ordinary differential equations, each point in \( F_{\gamma(t)} \) that does not lie in \( \partial_h E \) has a neighborhood in \( p^{-1}(\gamma([a, b])) \) in which \( \gamma \) has horizontal lifts. Since the metric is a product near \( \partial_h E \) such that the \( I \)-fibers are vertical, the same is true for each point in \( \partial_h E \). Since the fiber is compact, for each \( t \) there exists an \( \epsilon \) such that for every \( x \in F_{\gamma(t)} \), the horizontal lift of \( \gamma \) through \( x \) exists for \( s \in (t - \epsilon, t + \epsilon) \), and the result follows using compactness of the interval \([a, b] \).

For the remainder of this section, assume that the metric on \( E \) is a product near \( \partial_h E \) such that the \( I \)-fibers of \( \partial_h E \times I \) are vertical. Each fiber \( F \) of \( E \) inherits a Riemannian metric from that of \( E \), and has an exponential map \( \exp_F \) which (where defined) carries vectors tangent to \( F \) to points of \( F \). Note that the path
§4. The Vertical and Aligned Exponentials

\( \text{Exp}_F(t\omega) \) is not generally a geodesic in \( E \). The vertical exponential \( \text{Exp}_v \) is defined by \( \text{Exp}_v(\omega) = \text{Exp}_F(\omega) \), where \( \omega \) is a vertical vector and \( F \) is the fiber containing \( \pi(\omega) \).

Before defining the aligned exponential map \( \text{Exp}_a \), we will motivate its definition. A vector field \( X : E \to T(E) \) is called aligned if \( p(x) = p(y) \) implies that \( p_*(X(x)) = p_*(X(y)) \). This happens precisely when there exist a vector field \( X_B \) on \( B \) and a vertical vector field \( X_V \) on \( E \) so that for all \( x \in E \),

\[
X(x) = (p_*|_{H_x})^{-1}(X_B(p(x))) + X_V(x).
\]

In particular, any vertical vector field is aligned. When \( X \) is aligned, the projected vector field \( p_*X \) is well-defined. The key property of \( \text{Exp}_a \) is that if \( X \) is an aligned vector field on \( E \), and \( \text{Exp}_a(X(x)) \) is defined for all \( x \), then the map of \( E \) defined by sending \( x \) to \( \text{Exp}_a(X(x)) \) will be fiber-preserving.

\( \text{Exp}_a \) is defined as follows. Consider a tangent vector \( \beta \) in \( B \) such that \( \text{Exp}(\beta) \) is defined. A geodesic segment \( \gamma_\beta \) starting at \( \pi(\beta) = \text{Exp}(t\beta), 0 \leq t \leq 1 \). Define \( \text{Exp}_a(\omega) \) to be the endpoint of the unique horizontal lift of \( \gamma_{p_*(\omega)} \) starting at \( \text{Exp}_v(\omega_v) \). Note that \( \text{Exp}_a(\omega) \) exists if and only if both \( \text{Exp}_v(\omega_v) \) and \( \text{Exp}(p_*(\omega)) \) exist. Clearly, when \( \text{Exp}_a(\omega) \) is defined, it lies in the fiber containing the endpoint of a lift of \( \gamma_{p_*(\omega)} \), and therefore \( p(\text{Exp}_a(\omega)) = \text{Exp}(p_*(\omega)) \). This immediately implies that if \( X \) is an aligned vector field on \( E \) such that \( \text{Exp}_a(X(x)) \) is defined for all \( x \in E \), then the map defined by sending \( x \) to \( \text{Exp}_a(X(x)) \) takes fibers to fibers, and in particular if \( X \) is vertical, it takes each fiber to itself.

In section 6 we will need a further refinement of the metric on \( E \), namely that it also be a product near \( \partial_v E \). To achieve this, we proceed as follows. If \( \partial_v E \) is empty, there is nothing needed, and if \( \partial_h E \) is empty, then we simply choose a metric which is a product near the boundary as in section 2. Assuming that both are nonempty, put \( Y = \partial(\partial_v E) = \partial(\partial_h E) = \partial_v E \cap \partial_h E \). Let \( R_h \) be the complete metric on \( \partial_h E \) that was used to construct the metric \( R \) on \( E \) that is a product on a collar \( \partial_h E \times [0,1]_1 \), where the subscript will distinguish this interval from another to be selected later. Since the choice of \( R_h \) was arbitrary, we may assume that \( R_h \) was a product near \( Y \). That is, after reparametrizing, we may assume that there is a collar \( Y \times [0,2]_2 \) of \( Y \) in \( \partial_h E \) such that \( R_h \) is a product on all of \( Y \times [0,2]_2 \). Now \( Y \times [0,2]_1 \) is a collar of \( Y \) in \( \partial_v E \), and \( Y \times [0,2]_1 \times [0,2]_2 \) is a partial collar of \( \partial_v E \) defined on the subset \( Y \times [0,2]_1 \). Extend this to a collar \( \partial_v E \times [0,2]_2 \). Let \( R_v \) be the product of the restricted
metric $R|_{\partial E}$ and the standard metric on $[0,2]_2$. Since $R_h$ was a product near $Y$, we have $R_v = R$ on $Y \times [0,1]_1 \times [0,2]_2$. Now form a new metric on $E$ by piecing together $R_v$ and $R$ using a partition of unity subordinate to the open cover $\{ \partial_v E \times [0,2]_2, E - \partial_v E \times [0,1]_2 \}$. On points of $Y \times [0,1]_1 \times [0,2]_2$ the new metric is just a linear combination of the form $tR + (1 - t)R$, so agrees with $R$. The resulting metric is both a product near $\partial_v E$ and a product near $\partial_v E$.

## §5. Projection of fiber-preserving diffeomorphisms

Throughout this section and the next, it is understood that $p: E \to B$ is a locally trivial smooth map as in section 4, such that the metric on $B$ is a product near $\partial B$, and the metric on $E$ is a product near $\partial E$ such that the $I$-fibers of $\partial_E E \times I$ are vertical. When $W$ is a vertical submanifold of $E$, it is automatic that $W$ meets the collar $\partial_E E \times I$ in $I$-fibers, and we by rechoosing the metric on $B$ we may assume that $p(W)$ meets the collar $\partial B \times I$ in $I$-fibers.

Define $\partial_v W = W \cap \partial_E E$ and $\partial_v W = W \cap \partial_v E$.

Define $\mathcal{A}(W,T(E))$ to be the sections $X$ from $W$ to $T(E)|_W$ such that

1. $X$ is aligned, that is, if $p(w_1) = p(w_2)$ then $p_*(X(w_1)) = p_*(X(w_2))$,

2. if $x \in \partial_v W$, then $X(x)$ is tangent to $\partial_v E$, and if $x \in \partial_v W$, then $X(x)$ is tangent to $\partial_v E$, and

3. $\text{Exp}_a(X(x))$ is defined for all $x \in W$.

When $W = E$, the vector fields satisfying (1), (2), and (3) are denoted by $\mathcal{A}(T(E))$. The embellishments $\mathcal{A}^L(T(E))$ and $\mathcal{A}_{<1/2}(W,T(E))$ have the same meanings as in section 3. The elements of $\mathcal{A}(W,T(E))$ such that $p_*(X(x)) = Z(p(x))$ for all $x \in W$ are denoted by $\mathcal{V}(W,T(E))$, and similarly for $\mathcal{V}(T(E))$.

Define $F_a: \mathcal{A}^L(T(E)) \to \text{Maps}^L(E,E)$ by $F_a(X)(x) = \text{Exp}_a(X(x))$. Since $\text{Exp}_a$ is smooth, it follows as in lemma a of [10] that $F_a$ is continuous.

**Theorem 5.1** Let $K$ be a compact subset of $B$. Let $S$ be a subset of $\partial B$, and let $T = p^{-1}(S)$. Then $\text{Diff}^K(B \text{ rel } S)$ admits local $\text{Diff}^p_{\text{rel } T}(E \text{ rel } T)$ cross-sections.

**Proof of 5.1**: Choose a compact subset $L$ of $B$ such that $K \subseteq \text{int}(L)$. By lemma 3.5, there exist $\delta > 0$ and a continuous map $X_1: \text{Imb}_{<\delta}(L,B) \to$
$X \subseteq (L,T(B))$ such that $\text{Exp}(X_1(j)(x)) = j(x)$ for all $x \in L$ and all $j \in \text{Imb}_{<\delta}(L,B)$. Moreover, if $j(x) = x$, then $X_1(j)(x) = Z(x)$.

Let $\rho: \text{Diff}^K(B) \to \text{Imb}(L,B)$ be restriction. Put $U_0 = \rho^{-1}(\text{Imb}_{<\delta}(L,B))$, a neighborhood of $1_B$, and define $X_0: U_0 \to X(T(B))$ by

$$X_0(f)(x) = \begin{cases} X_1(f|_L)(x) & \text{if } x \in L \\ Z(x) & \text{if } x \notin K. \end{cases}$$

This makes sense since if $x \in L - K$, then $f|_L(x) = x$ so $X_1(f|_L)(x) = Z(x)$. We have $X_0(1_B) = Z$ and $\text{Exp}(X_0(f)(x)) = f(x)$ for every $f \in U_0$ and $x \in B$.

Let $h \in \text{Diff}^K(B)$. For every $g \in U_0h$, $\text{Exp}(X_0(gh^{-1}(x))) = gh^{-1}(x)$. Define

$$\tilde{\chi}(g)(x) = (p_*|_{\mathcal{H}})^{-1}(X_0(gh^{-1})(p(x))),$$

so that $\tilde{\chi}(g)$ is an aligned section of $T(E)$. We have that $\text{Exp}_a(\tilde{\chi}(g)(x))$ exists since $\text{Exp}(p_*\tilde{\chi}(g)(x)) = \text{Exp}(X_0(gh^{-1})(p(x))) = gh^{-1}(p(x))$ exists and $\text{Exp}(\tilde{\chi}(g)(x)) = x$ exists. The other conditions are easily checked to verify that $\tilde{\chi}(g) \in \mathcal{A}^{p^{-1}(K)}(T(E))$.

Let $J$ be a neighborhood of $1_M \in \text{Maps}^{p^{-1}(K)}(E,E)$ given by lemma 3.3, and put $U = \tilde{\chi}^{-1}F_0^{-1}(J)$. Define $\chi: U \to \text{Diff}_J(E)$ by $\chi(g) = F_0\tilde{\chi}(g)$. The local cross-section condition holds, since given $b \in B$ we may choose $x$ with $p(x) = h(b)$ and calculate that

$$\overline{\chi(g)}h(b) = p(\chi(g)(x))$$

$$= p(\text{Exp}_a(\tilde{\chi}(g)(x)))$$

$$= \text{Exp}(X_0(gh^{-1})(h(b)))$$

$$= gh^{-1}(h(b))$$

If $g, h \in \text{Diff}^K(B \text{ rel } S)$, then $X_0(gh^{-1})(x) = Z(x)$ for all $x \in \partial S$. It follows that $\tilde{\chi}(g)(x) = Z(x)$ for all $x \in T$, so $\chi(g) \in \text{Diff}_{f}^{p^{-1}(K)}(E \text{ rel } T)$.

From proposition 3.1, we have immediately

**Theorem 5.2** Let $K$ be a compact subset of $B$. Let $S \subseteq \partial B$ and let $T = p^{-1}(T)$. Then $\text{Diff}_{f}^{p^{-1}(K)}(E \text{ rel } T) \to \text{Diff}^K(B \text{ rel } S)$ is locally trivial.
The homotopy extension property of the fibration in theorem 5.2 yields immediately the following corollary. As indicated in the introduction, each of the fibration theorems we prove in this paper has a corresponding corollary involving parameterized lifting or extension, but since the statements are all analogous we give only the following one as a prototype.

**Corollary 5.3** (Parameterized Isotopy Extension Theorem) Let \( K \) be a compact subset of \( B \), let \( S \subseteq \partial B \), and let \( T = p^{-1}(S) \). Suppose that for each \( t \) in a path-connected parameter space \( P \) there is an isotopy \( g_{t,s} \), which is the identity on \( S \) and outside of \( K \), such that \( g_{t,0} \) lifts to a diffeomorphism \( G_{t,0} \) of \( E \) which is the identity on \( T \). Assume that sending \( (t,s) \rightarrow g_{t,s} \) defines a continuous function from \( P \times [0,1] \) to \( \text{Diff}(B \text{ rel } S) \) and sending \( t \) to \( G_{t,0} \) defines a continuous function from \( P \) to \( \text{Diff}(E \text{ rel } T) \). Then the family \( G_{t,0} \) extends to a continuous family on \( P \times I \) such that for each \( (t,s) \), \( G_{t,s} \) is a fiber-preserving diffeomorphism inducing \( g_{t,s} \) on \( B \).

### 6 Restriction of fiber-preserving diffeomorphisms

In this section we present the analogues of the main results of [10] in the fibered case. As in section 5, we assume that the metric on \( B \) is a product near \( \partial B \), and the metric on \( E \) is a product near \( \partial v E \) such that the \( I \)-fibers of \( \partial v E \times I \) are vertical. We further assume that the metric on \( E \) is a product near \( \partial v E \); this is needed only in the proof of lemma 6.1.

It is first necessary to adapt lemmas 3.4 and 3.5.

**Lemma 6.1** Let \( W \) be a compact vertical submanifold of \( E \). Let \( T \) be a closed fibered neighborhood in \( \partial v E \cap \partial v W \), and let \( L \subseteq E \) be a neighborhood of \( W \). For sufficiently small \( \delta \) there exists a continuous map \( k: A_{<\delta}(W,T(E)) \rightarrow A^L(T(E)) \) such that \( k(X)(x) = X(x) \) for all \( x \in W \) and \( X \in A_{<\delta}(W,T(E)) \). If \( X(x) = Z(x) \) for all \( x \in T \cap \partial v W \), then \( k(X)(x) = Z(x) \) for all \( x \in T \). Furthermore, \( k(V_{<\delta}(W,T(E))) \subset V^L(T(E)) \).

**Proof of 6.1:** Assume first that \( W \) has positive codimension. Since the fiber of \( p \) is compact, we may assume that \( p(L) \) is a neighborhood of \( p(W) \) with \( p^{-1}(p(L)) = L \). Since \( W \) is compact we may choose \( \delta < 1/2 \) such that if \( X \in A_{<\delta}(W,T(E)) \) then \( \| p_* X(x) \| < 1/2 \) for all \( x \in p(W) \). Let \( k_B: X_{<1/2}(p(W),T(B)) \rightarrow X^{p(l)}(T(B)) \) be given by lemma 3.4, using \( p(T) \) as
the neighborhood $S$ in lemma 3.4. The vectors $(p_*|_{H_a})^{-1}(k_B(p_*X)(p(x)))$ will give the horizontal part of our extension $k$, but to obtain sufficient control of the vertical part we will need to adapt the proof of lemma 3.4 using $\text{Exp}_a$.

Let $\nu_\epsilon(W)$ be the $\epsilon$-normal bundle of $W$. Note that its fibers are horizontal, since the tangent space of $W$ includes the bundle of vertical vectors of $T(E)$ at points of $W$. For sufficiently small $\epsilon$, $j_\epsilon \cdot \nu_\epsilon(W) \to E$ can be defined by $j_\epsilon(\omega) = \text{Exp}_a(\omega)$ and carries $\nu_\epsilon(W)$ diffeomorphically to a neighborhood of $W$ in $E$. We choose $\epsilon$ small enough so that this neighborhood is contained in $L$, and so that the image of the fibers at points of $(\partial_v E - T) \cap \partial_r W$ lies in $T$ and the image of the fibers at points of $(\partial_v E - T) \cap \partial_r W$ lies in $\partial_r E - T$.

If $x \in \partial_r E \times \{t\}$, $j_\epsilon$ carries the normal fiber at $x$ into $\partial_r E \times \{t\}$. Since $W$ is compact, we may choose $\epsilon$ small enough so that $j_\epsilon(\omega) \in \partial_r E \times I$ only when $\pi(\omega) \in \partial_r E \times I$.

Suppose $v \in T_x(E)$ and that $\text{Exp}_a(v)$ is defined. For all $u \in T_x(E)$ define $P_a(u, v)$ to be the vector that results from parallel translation of $u$ along the path that sends $t$ to $\text{Exp}_a(tv)$, $0 \leq t \leq 1$. Let $\alpha: E \to [0, 1]$ be a smooth function which is identically 1 on $W$ and identically 0 on $E - j_\epsilon(\nu_{\epsilon/2}(W))$. Define $k_E: \mathcal{A}_{<\delta}(W, T(E)) \to \mathcal{V}^k(T(E))$ by

$$k_E(X)(x) = \begin{cases} 
\alpha(x)P_a(X(\pi(j_\epsilon^{-1}(x)))j_\epsilon^{-1}(x) \right) & \text{for } x \in j_\epsilon(\nu_\epsilon(W)) \\
Z(x) & \text{for } x \in E - j_\epsilon(\nu_{\epsilon/2}(W))
\end{cases}.$$ 

Note that if $X$ is vertical, then $k(X)(x) = X(x)$ for all $x \in W$. For later use we make two observations about $k_E$. First, if $X(x) = 0$ for all $x \in T \cap \partial_r W$, then $k_E(X)(x) = 0$ for all $x \in T$. This is because $j_\epsilon^{-1}(T)$ consists exactly of the vectors normal to $W$ at the points of $T \cap \partial_r W$. Second, if $x = (y, t) \in \partial_r E \times I$, and $x \in j_\epsilon(\nu_\epsilon(W))$, then $\pi(j_\epsilon^{-1}(x))$ is of the form $(y', t)$, and either $k_E(X)(x) = Z(x)$ or the component of $k_E(X)(x)$ in the $I$-direction is of the form $\beta \omega_I$, where $0 < \beta < 1$ and $\omega_I$ is the component of $X(\pi(j_\epsilon^{-1}(x)))$ in the $I$-direction. This follows because the metric is a product on $\partial_r E \times I$, so parallel translation preserves the component in the $I$-direction. Consequently, since $\text{Exp}_a(X(\pi(j_\epsilon^{-1}(x))))$ is defined, so is $\text{Exp}_a(k(X)(x))$.

For $X \in \mathcal{A}_{<\delta}(W, T(E))$, define $X_v(x) = X(x)_v$, and put

$$k(X)(x) = (p_*|_{H_a})^{-1}(k_B(p_*X)(p(x))) + k_E(X_v)(x).$$

We need to check that $k(X)$ lies in $\mathcal{A}^k(T(E))$. From its definition, $k(X)$ is aligned and vanishes outside of $L$. Let $x \in \partial_r E$ and suppose that $k(X)(x) \neq 0$. Then
Z(x). Then \( \pi(j^{-1}_a(x)) \in \partial hE \) and since \( X \in \mathcal{A}(W, T(E)) \), \( X(\pi(j^{-1}_a(x))) \) is tangent to \( \partial hE \). Since the metric is a product near \( \partial hE \), parallel translation preserves vectors tangent to \( \partial hE \), and it follows that \( k(X)(x) \) is tangent to \( \partial hE \).

Suppose that \( x \in \partial hE \) and \( k_t(X)(x) \neq Z(x) \). Since \( k_B(p_\ast X)(p(x)) \) is tangent to \( \partial B \), \( (p_\ast|_{H_\epsilon})^{-1}(k_B(p_\ast X)(p(x))) \) is tangent to \( \partial B \). The fact that the metric on \( E \) is a product near \( \partial eE \) implies that \( \pi(j^{-1}_a(x)) \in \partial eW \), and moreover, since \( X(\pi(j^{-1}_a(x))) \) is tangent to \( \partial eE \), that \( P_a(X(\pi(j^{-1}_a(x))), j^{-1}_a(x)) \) is also tangent to \( \partial eE \). We conclude that \( k(X)(x) \) is tangent to \( \partial eE \). The fact that \( \text{Exp}(k_B(p_\ast X)(p(x))) \) was defined, together with the second observation after the definition of \( k_E \), implies that \( \text{Exp}(k(X)(x)) \) is always defined.

Suppose that \( X(x) = Z(x) \) for all \( x \in T \cap \partial hE \). Then \( p_\ast(X)(p(x)) = Z(p(x)) \) for all \( x \in p(T) \cap \partial (p(W)) \), so \( k_B(p_\ast X)(p(x)) = Z(p(x)) \) for all \( x \in p(T) \). The first observation after the definition of \( k_E \) shows that \( k_E(X)(x) = Z(x) \) for all \( x \in T \). Therefore \( k(X)(x) = Z(x) \) for all \( x \in T \).

For the last statement, if \( X \in \mathcal{V}(W, T(E)) \), then \( p_\ast(X)(p(x)) = Z(p(x)) \) for all \( x \in \partial (p(W)) \), so \( k_B(p_\ast X)(p(x)) = Z(p(x)) \) for all \( x \in \partial eE \). Therefore \( k(X) \in \mathcal{V}(T(E)) \).

The case when \( W \) has codimension zero is similar. As in the proof of lemma 3.4, use the subset \( \nu^+_e Fr(W) \) consisting of the vectors in the normal bundle of the frontier of \( W \) whose aligned exponential lies in \( E - W \), and define

\[
k_E(X)(x) = \begin{cases} X(x) & \text{for } x \in W \\ \alpha(x) P_a(X(\pi(j^{-1}_a(x))), j^{-1}_a(x))_v & \text{for } x \in j_a(\nu^+_e Fr(W)) \\ Z(x) & \text{for } x \in E - j_a(\nu^+_e Fr(W)) \end{cases}
\]

For the next lemma we will adapt the neighborhood \( N_e(V) \) used in the proof of lemma 3.5 into the fibered context. For \( x \in E \), let \( R_B(p(x), \epsilon) \) be the subset of \( T_{p(x)}(B) \) as defined before the statement of lemma 3.5. Denote \( p^{-1}(p(x)) \) by \( F \), and let \( R_F(x, \epsilon) \) be the subset of \( T_x(F) \) defined before the statement of lemma 3.5. Regard \( T_x(F) \) as the vertical subset \( V_x(E) \) of \( T_x(E) \), and observe that for sufficiently small \( \epsilon \), the aligned exponential \( \text{Exp}_a \) carries \( R_F(x, \epsilon) \) to a neighborhood of \( x \) in \( R \), since on \( T_x(F) \), \( \text{Exp}_a \) agrees with the exponential map of \( F \). Now define \( S(x, \epsilon) = R_F(x, \epsilon) \times (p_\ast|_{H_\epsilon})^{-1}(R_B(p(x), \epsilon)) \subset \)
$V_x(E) \times H_x(E) = T_x(E)$. For a vertical submanifold $W \subseteq E$, define $N_\epsilon(W) = \bigcup_{x \in W} S(x, \epsilon)$. Provided that $W$ is compact, we may choose a positive $\epsilon$ such that for each $x \in W$, $\text{Exp}_x: N_\epsilon(W) \cap T_x(E) \to E$ is a diffeomorphism onto a neighborhood of $x$ in $E$.

**Lemma 6.2** Let $W$ be a compact vertical submanifold of $E$. For sufficiently small $\delta$, there exists a continuous map $X: (\text{Imb}_f)_{<\delta}(W, E) \to \mathcal{A}(W, T(E))$ such that $\text{Exp}_a(X(j)(x)) = j(x)$ for all $x \in W$ and $j \in (\text{Imb}_f)_{<\delta}(W, E)$, and moreover if $j(x) = i_W(x)$ then $X(j)(x) = Z(x)$. Furthermore, $X((\text{Imb}_v)_{<\delta}(W, E)) \subseteq \mathcal{V}(W, T(E))$.

**Proof of 6.2:** Let $N_\epsilon(W)$ be as defined above, with $\epsilon$ small enough to ensure the local diffeomorphism condition. Choose $\delta$ small enough so that for every $x \in W$ and every $j \in (\text{Imb}_f)_{<\delta}(W, E)$, $j(x) \in \text{Exp}_a(N_\epsilon(W) \cap T_x(E))$. Define $X(j)(x)$ to be the unique vector in $N_\epsilon(W) \cap T_x(M)$ such that $\text{Exp}_a(X(j)(x))$ equals $j(x)$.

In the statements of our remaining results, the notation $\text{Imb}(W, E \text{ rel } T)$ is as defined before the statement of theorem 3.6.

**Theorem 6.3** Let $W$ be a compact vertical submanifold of $E$. Let $T$ be a closed fibered neighborhood in $\partial_x E$ of $T \cap \partial_x W$, and let $L$ be a neighborhood of $W$. Then

(i) $\text{Imb}^L_f(W, E \text{ rel } T)$ admits local $\text{Diff}^L_f(E \text{ rel } T)$ cross-sections, and

(ii) $\text{Imb}^L_v(W, E \text{ rel } T)$ admits local $\text{Diff}^L_v(E \text{ rel } T)$ cross-sections.

**Proof of 6.3:** By proposition 3.2, it suffices to find local cross-sections at the inclusion $i_W$. Choose a compact neighborhood $K$ of $W$ with $K \subseteq L$. Let $k: \mathcal{A}_{<\delta}(W, T(E)) \to \mathcal{X}^K(T(E))$ be obtained using lemma 6.1. Using lemma 6.2, choose $\delta_1 > 0$ and $X: (\text{Imb}_f)_{<\delta_1}(W, E) \to \mathcal{A}(W, T(E))$. If $j \in \text{Imb}^L_f(W, E \text{ rel } T)$, then $X(j)(x) = 0$ for all $x \in \partial_v E$. Since $X$ is continuous and $X(i_W) = Z$, we may choose a neighborhood $U$ of $i_W$ in $\text{Imb}^L_f(W, E \text{ rel } T)$ so that $X(U) \subset \mathcal{A}_{<\delta}(W, T(E))$. For $j \in U$ define $\chi(j) = F_{\text{ad}} X(j)$. From lemma 3.3, we may make $U$ small enough to ensure that $\chi(j)$ is a diffeomorphism, and then $\chi$ is the cross-section that proves (a). For (b), suppose that
6. Restriction of fiber-preserving diffeomorphisms

Since \( k(V_c \triangleleft (W, T(E))) \subseteq \mathcal{V}^K(T(E)) \), \( \chi_j \) lies in \( \text{Diff}_v(E \text{ rel } T) \), so the restriction of \( \chi \) to \( U \cap \text{Imb}_v^L(W, E \text{ rel } T) \) is the necessary cross-section.

Using proposition 3.1, we have the following immediate corollaries.

**Corollary 6.4** Let \( W \) be a compact vertical submanifold of \( E \). Let \( T \) be a closed fibered neighborhood in \( \partial_v E \) of \( T \cap \partial_v W \), and \( L \) a neighborhood of \( W \). Then the following restrictions are locally trivial:

(i) \( \text{Diff}_f^L(E \text{ rel } T) \to \text{Imb}_f^L(W, E \text{ rel } T) \), and

(ii) \( \text{Diff}_v^L(E \text{ rel } T) \to \text{Imb}_v(W, E \text{ rel } T) \).

**Corollary 6.5** Let \( V \) and \( W \) be vertical submanifolds of \( E \). Let \( T \) be a closed fibered neighborhood in \( \partial_v E \) of \( T \cap \partial_v V \), and let \( L \) a neighborhood of \( V \). Then the following restrictions are locally trivial:

(i) \( \text{Imb}_f^L(V, E \text{ rel } T) \to \text{Imb}_f^L(W, E \text{ rel } T) \).

(ii) \( \text{Imb}_v(V, E \text{ rel } T) \to \text{Imb}_v(W, E \text{ rel } T) \).

The final result of this section includes some of our previous results.

**Theorem 6.6** Let \( W \) be a compact vertical submanifold of \( E \). Let \( K \) be a compact neighborhood of \( p(W) \) in \( B \). Let \( T \) be a closed fibered neighborhood in \( \partial_v \Sigma \) of \( T \cap \partial_v W \), and put \( S = p(T) \). Then all four maps in the following square are locally trivial:

\[
\begin{array}{ccc}
\text{Diff}_{-1}^{p^{-1}(K)}(E \text{ rel } T) & \to & \text{Imb}_{-1}^{p^{-1}(K)}(W, E \text{ rel } T) \\
\downarrow & & \downarrow \\
\text{Diff}^K(B \text{ rel } S) & \to & \text{Imb}^K(p(W), B \text{ rel } S) .
\end{array}
\]
§7. Palais’ theorem for orbifolds

Proof of 6.6: The top arrow is corollary 6.4(i), the left vertical arrow is theorem 5.1, and the bottom arrow is corollary 3.7. For the right vertical arrow, we will first show that \( \text{Imb}^K_p(p(W), B \text{ rel } S) \) admits local \( \text{Diff}^{p-1(K)}_f(E \text{ rel } T) \) cross-sections. Let \( i \in \text{Imb}^K_p(p(W), B \text{ rel } S) \). Using theorems 6.3 and 5.1, choose local cross-sections \( \chi_1: U \to \text{Diff}^K(B \text{ rel } S) \) at \( i \) and \( \chi_2: V \to \text{Diff}^{p-1(K)}_f(E \text{ rel } T) \) at \( \chi_1(i) \). Let \( U_1 = \chi_1^{-1}(V) \), then for \( j \in U_1 \) we have

\[
(\chi_2 \chi_1(j))i = \chi_2(\chi_1(j))i = \chi_1(j)i = j.
\]

Since the right vertical arrow is \( \text{Diff}^{p-1(K)}_f(E \text{ rel } T) \)-equivariant, proposition 3.1 implies it is locally trivial.

7 Palais’ theorem for orbifolds

In this section, we prove the main results from [10] in the context of orbifolds. Let \( \mathcal{O} \) be a connected smooth orbifold whose universal covering \( \tilde{\mathcal{O}} \) is a manifold. Denote by \( \tau: \tilde{\mathcal{O}} \to \mathcal{O} \) the orbifold universal covering, and let \( H \) be its group of covering transformations. Since \( \mathcal{O} \) is smooth, the elements of \( H \) are diffeomorphisms.

Let \( \text{Maps}_H(\tilde{\mathcal{O}}, \tilde{\mathcal{O}}) \) be the space of weakly \( H \)-equivariant maps, that is, the maps \( f: \tilde{\mathcal{O}} \to \tilde{\mathcal{O}} \) such that for some automorphism \( \alpha \) of \( H \), \( f(h(x)) = \alpha(h)(f(x)) \) for all \( x \in \tilde{\mathcal{O}} \) and \( h \in H \). Let \( \text{Diff}_H(\tilde{\mathcal{O}}) \) be the weakly \( H \)-equivariant diffeomorphisms of \( \tilde{\mathcal{O}} \). It is the normalizer of \( H \) in \( \text{Diff}(\tilde{\mathcal{O}}) \).

An orbifold diffeomorphism of \( \mathcal{O} \) is by definition an orbifold homeomorphism of \( \mathcal{O} \) whose lifts to \( \tilde{\mathcal{O}} \) are diffeomorphisms. Thus the group \( \text{Diff}(\mathcal{O}) \) of orbifold diffeomorphisms of \( \mathcal{O} \) is the quotient of the group \( \text{Diff}_H(\tilde{\mathcal{O}}) \) by the normal subgroup \( H \).

Throughout this section, we let \( W \) be a compact suborbifold of \( \mathcal{O} \). By definition, the preimage \( \tilde{W} \) in \( \tilde{\mathcal{O}} \) is a submanifold, and the space of orbifold imbeddings \( \text{Imb}(W, \mathcal{O}) \) can be regarded as the quotient of \( \text{Imb}_H(\tilde{W}, \tilde{\mathcal{O}}) \) by the action of \( H \). For spaces of vectors, a subscript \( H \) will indicate the equivariant ones, thus for example \( \chi_H(\tilde{W}, T(\tilde{\mathcal{O}})) \) means the \( H \)-equivariant sections of the restriction of \( T(\tilde{\mathcal{O}}) \) to \( \tilde{W} \), satisfying conditions (1) and (2) given in section 3.
Provided that $H$ acts as isometries on the $H$-invariant subset $\tilde{L}$ of $\tilde{O}$, the evaluation map $F$ carries $\mathcal{X}_H^H(T(\tilde{O}))$ into $\text{Maps}_{\tilde{H}}^{\tilde{L}}(\tilde{O}, \tilde{O})$.

The next two lemmas provide equivariant functions and metrics.

**Lemma 7.1** Let $H$ be a group acting smoothly and properly discontinuously on a manifold $M$, possibly with boundary, such that $M/H$ is compact. Let $A$ be an $H$-invariant closed subset of $M$, and $U$ an $H$-invariant neighborhood of $A$. Then there exists an $H$-equivariant smooth function $\gamma: M \to [0, 1]$ which is identically equal to 1 on $A$ and whose support is contained in $U$.

**Proof of 7.1:** Fix a compact subset $C$ of $M$ which maps surjectively onto $M/H$ under the quotient map. Let $\phi: M \to [0, \infty)$ be a smooth function such that $\phi(x) \geq 1$ for all $x \in C \cap A$ and whose support is compact and contained in $U$. Define $\psi$ by $\psi(x) = \sum_{h \in H} \phi(h(x))$. Now choose $\eta: \mathbb{R} \to [0, 1]$ such that $\eta(r) = 0$ for $r \leq 0$ and $\eta(r) = 1$ for $r \geq 1$, and put $\gamma = \eta \circ \psi$.

**Lemma 7.2** Let $H$ be a group acting smoothly and properly discontinuously on a manifold $M$, possibly with boundary, such that $M/H$ is compact. Let $N$ be a properly imbedded $H$-invariant submanifold, possibly empty. Then $M$ admits a complete Riemannian metric, which is a product near $\partial M$ and such that $N$ meets the collar $\partial M \times I$ in $I$-fibers, such that $H$ acts as isometries. Moreover, the action preserves the collar, and if $(y, t) \in \partial M \times I$ and $h \in H$, then $h(y, t) = (h|_{\partial M}(y), t)$.

**Proof of 7.2:** We first prove that equivariant Riemannian metrics exist. Choose a compact subset $C$ of $M$ that maps surjectively onto $M/H$ under the quotient map. Let $\phi: M \to [0, \infty)$ be a compactly supported smooth function which is positive on $C$. Choose a Riemannian metric $R$ on $M$ and denote by $R_x$ the inner product which $R$ assigns to $T_x(M)$. Define a new metric $R'$ by

$$R'_x(v, w) = \sum_{h \in H} \phi(h(x)) R_{h(x)}(h_*(v), h_*(w)).$$
Since $\phi$ is compactly supported, the sum is finite, and since every orbit meets the support of $\phi$, $R'$ is positive definite. To check equivariance, let $g \in H$. Then

$$R'_{g(x)}(g_*(v), g_*(w)) = \sum_{h \in H} \phi(h(g(x))) R_{h(g(x))}(h_*(g_*(v)), h_*(g_*(w)))$$

$$= \sum_{h \in H} \phi(hg(x)) R_{hg(x)}((hg)_*(v), (hg)_*(w))$$

$$= R'_x(v, w).$$

We need to improve the metric near the boundary. First, note that $C \cap \partial M$ maps surjectively onto the image of $\partial M$. Choose an inward-pointing vector field $\tau'$ on a neighborhood $U$ of $C \cap \partial M$, which is tangent to $N$. Choose a smooth function $\phi: M \to [0, \infty)$ which is positive on $C \cap \partial M$ and has support contained in $U$. The field $\phi \tau'$ defined on $U$ extends using the zero vector field on $M - U$ to a vector field $\tau$ which is nonvanishing on $C \cap \partial M$. For $x$ in the union of the $H$-translates of $U$, define $\omega_x = \sum_{h \in H} \phi(h(x)) h^{-1}_*(\tau_{h(x)})$. This is defined, nonsingular, and equivariant on an equivariant neighborhood of $\partial M$, and we use it to define a collar $\partial M \times [0, 2]$ equivariant in the sense that if $(y, t) \in \partial M \times [0, 2]$ then $h(y, t) = (h|_{\partial M}(y), t)$. Moreover, $N$ meets this collar in $I$-fibers. On $\partial M \times [0, 2]$, choose an equivariant metric $R_1$ which is the product of an equivariant metric on $\partial M$ and the standard metric on $[0, 2]$, and choose any equivariant metric $R_2$ defined on all of $M$. Using lemma 7.1, choose $H$-equivariant functions $\phi_1$ and $\phi_2$ from $M$ to $[0, 1]$ so that $\phi_1(x) = 1$ for all $x \in \partial M \times [0, 3/2]$ and the support of $\phi_1$ is contained in $\partial M \times [0, 2]$, and so that $\phi_2(x) = 1$ for $x \in M - \partial M \times [0, 3/2]$ and the support of $\phi_2$ is contained in $M - \partial M \times [0, 1]$. Then, $\phi_1 R_1 + \phi_2 R_2$ is $H$-equivariant and is a product near $\partial M$, and $N$ is vertical in $\partial M \times I$.

Since $M/H$ is compact and $H$ acts as isometries, the metric must be complete. For let $C$ be a compact subset of $M$ that maps surjectively onto $M/H$. We may enlarge $C$ to a compact codimension-zero submanifold $C'$ such that every point of $M$ has a translate which lies in $C'$ at distance at least a fixed $\epsilon$ from the frontier of $C'$. Then, any Cauchy sequence in $M$ can be translated, except for finitely many terms, into a Cauchy sequence in $C'$. Since $C'$ is compact, this converges, so the original sequence also converged.
We need the equivariant analogue of lemma 3.3. Its proof uses the following general fact.

**Proposition 7.3** Suppose that $H$ acts properly discontinuously on a locally compact connected space $X$, and that $X/H$ is compact. Then $H$ is finitely generated.

**Proof of 7.3:** Using local compactness, there exists a compact set $C$ which maps surjectively to $X/H$. Let $H_0$ be the subgroup generated by the finitely many elements $h$ such that $h(C) \cap C$ is nonempty. The union of the $H_0$-translates of $C$ is an open and closed subset, so must equal $X$. This implies that $H = H_0$.

**Lemma 7.4** Let $\tilde{K}$ be an $H$-invariant subset of $\tilde{O}$ whose quotient in $O$ is compact. Then there exists a neighborhood $J$ of $1_{\tilde{O}}$ in $\text{Maps}^H_\tilde{K}(\tilde{O}, \tilde{O})$ that consists of diffeomorphisms.

**Proof of 7.4:** Suppose first that $O$ is compact. Then by proposition 7.3, $H$ is finitely generated. We claim that if $f$ is a map that is close enough to $1_{\tilde{O}}$, then $f$ commutes with the $H$-action. Choose an $x \in \tilde{O}$ which is not fixed by any nontrivial element of $H$. Define $\Phi: \text{Maps}^H_\tilde{K}(\tilde{O}, \tilde{O}) \to \text{End}(H)$ by sending $f$ to $\phi_f$ where $f(h(x)) = \phi_f(h)f(x)$. This is independent of the choice of $x$, hence is a homomorphism. If $f$ is close enough to $1_{\tilde{O}}$ on $\{x, h_1(x), \ldots, h_n(x)\}$, where $\{h_1, \ldots, h_n\}$ generates $H$, then $\phi_f = 1_H$. This prove the claim.

Next we show that for $f$ close enough to $1_{\tilde{O}}$, $f^{-1}(S)$ is compact whenever $S$ is compact. From above, we may assume that $f$ commutes with the $H$-action. Let $C$ be a compact set in $\tilde{O}$ which maps surjectively to $O$. If $S$ is a set for which $f^{-1}(S)$ meets infinitely many translates of $C$ then so does $S$, and $S$ could not be compact.

Consider $f$ close enough to $1_{\tilde{O}}$ to ensure the previous conditions. By requiring $f$ sufficiently $C^\infty$-close to $1_{\tilde{O}}$, $f_*$ is nonsingular at each point of $C$, hence on all of $\tilde{O}$. If follows that $f$ is a local diffeomorphism. Since also $f$ takes boundary to boundary and preimages of compact sets are compact, $f$ is a covering map. Since $\tilde{O}$ is simply-connected, $f$ is a diffeomorphism.
§7. Palais’ theorem for orbifolds

Now suppose that \( \mathcal{O} \) is noncompact. Enlarge \( \overline{K}/H \) to a compact codimension-zero suborbifold \( L \). Let \( \tilde{L}' \) be a single component of \( \tilde{L} \) and \( H' \) the stabilizer of \( \tilde{L}' \) in \( H \). Let \( f \) be a covering map of \( \tilde{L}' = \tilde{L} \) and by the previous argument we may assume that \( f \) is a covering map on \( \tilde{L}' \) (although since we don’t know that \( \tilde{L}' \) is simply connected, we cannot immediately conclude that \( f \) is a diffeomorphism). Since \( \mathcal{O} \) is connected and noncompact, \( L \) has frontier in \( \mathcal{O} \), hence \( \tilde{L}' \) has frontier in \( \tilde{\mathcal{O}} \). Since \( f \) is the identity on \( \mathcal{O} – \overline{K} \), \( f \) must be a diffeomorphism on \( \tilde{L}' \), hence on all of \( \tilde{L} \), hence on all of \( \tilde{\mathcal{O}} \).

We now prove the analogues of lemmas 3.4 and 3.5 for vector fields on \( \mathcal{O} \). Assume that \( \mathcal{W} \) is a compact suborbifold of \( \mathcal{O} \).

**Lemma 7.5** Let \( \mathcal{W} \) be a compact suborbifold of \( \mathcal{O} \). Let \( L \) be a neighborhood of \( \mathcal{W} \) in \( \mathcal{O} \) and let \( \mathcal{S} \) be a closed neighborhood in \( \partial \mathcal{O} \) of \( S \cap \partial \mathcal{W} \). Denote the preimages in \( \tilde{\mathcal{O}} \) by \( \tilde{L} \) and \( \tilde{\mathcal{S}} \). Then there exists a continuous map \( k: (\mathcal{X}_H)_{<1/2}(\tilde{\mathcal{W}}, T(\tilde{\mathcal{O}})) \to \mathcal{X}_H(T(\tilde{\mathcal{O}})) \) such that \( k(X)(x) = X(x) \) for all \( x \) in \( \mathcal{W} \) and \( X \in (\mathcal{X}_H)_{<1/2}(\tilde{\mathcal{W}}, T(\tilde{\mathcal{O}})) \). Moreover, \( k(Z) = Z \), and if \( X(x) = Z(x) \) for all \( x \in \tilde{S} \cap \partial \mathcal{W} \), then \( k(X)(x) = Z(x) \) for all \( x \in \tilde{S} \).

**Proof of 7.5:** Assume first that \( \mathcal{W} \) has positive codimension. Replacing \( L \) by a compact orbifold neighborhood of \( \mathcal{W} \) and using lemma 7.2, we may assume that \( H \) acts as isometries on \( \tilde{\mathcal{O}} \), that the metric is a product near \( \partial \tilde{\mathcal{O}} \), and that \( \mathcal{W} \) meets the collar \( \partial \tilde{\mathcal{O}} \times I \) in \( I \)-fibers. Let \( \nu(\mathcal{W}) \) be the normal bundle, regarded as a subbundle of the restriction of \( T(\tilde{\mathcal{O}}) \) to \( \mathcal{W} \). For \( \epsilon > 0 \), let \( \nu_\epsilon(\mathcal{W}) \) be the subspace of all vectors of length less than \( \epsilon \). Since \( \mathcal{W} \) is compact and \( H \) acts as isometries on \( \tilde{L} \), \( \exp \) imbeds \( \nu_\epsilon(\mathcal{W}) \) as a tubular neighborhood of \( \mathcal{W} \) for sufficiently small \( \epsilon \). By choosing \( \epsilon \) small enough, we may assume that \( \exp(\nu_\epsilon(\mathcal{W})) \subset \tilde{L} \), that the fibers at points in \( \tilde{\mathcal{S}} \) map into \( \tilde{\mathcal{S}} \), and that the fibers at points in \( \partial \tilde{\mathcal{O}} – \tilde{\mathcal{S}} \) map into \( \partial \tilde{\mathcal{O}} – \tilde{\mathcal{S}} \).

Now use lemma 7.1 to choose an \( H \)-equivariant smooth function \( \alpha: \tilde{\mathcal{O}} \to [0, 1] \) which is identically equal to 1 on \( \tilde{\mathcal{W}} \) and has support in \( j(\nu_{1/2}(\tilde{\mathcal{W}})) \). The extension \( k(X) \) can now be defined exactly as in lemma 3.4. Note that since \( H \) acts as isometries, the parallel translation function \( P \) is \( H \)-equivariant, and the \( H \)-equivariance of \( k(X) \) follows easily.
The case when $W$ has codimension zero is similar, using $\nu_\epsilon^+(\tilde{W})$ as in the proof of lemma 3.4.

**Lemma 7.6** For all sufficiently small positive $\delta$, there exists a continuous map $X: (\text{Imb}_H)^{<\delta}(\tilde{W}, \tilde{O}) \to (\mathcal{X}_H)^{<1/2}(\tilde{W}, T(\tilde{O}))$ such that $\text{Exp}(X(x)) = j(x)$ for all $x \in \tilde{W}$ and $j \in (\text{Imb}_H)^{<\delta}(\tilde{W}, \tilde{O})$, and moreover if $j(x) = i_{\tilde{W}}(x)$ then $X(j)(x) = Z(x)$.

**Proof of 7.6:** Replacing $\mathcal{O}$ by a compact orbifold neighborhood of $\mathcal{W}$ and using lemma 7.2, we may assume that $H$ acts as isometries on $\tilde{O}$, that the metric is a product near $\partial \tilde{O}$, and that $\tilde{W}$ meets the collar $\partial \tilde{O} \times I$ in $I$-fibers. Let $N_\epsilon(\tilde{W})$ be defined exactly as in section 3. By compactness of $\mathcal{W}$, there exists a positive $\epsilon$ such that for every $x \in \tilde{W}$, $\text{Exp}: N_\epsilon(\tilde{W}) \cap T_x(\tilde{O}) \to \tilde{O}$ is a diffeomorphism to an open neighborhood of $x$ in $\tilde{O}$, contained in $\tilde{L}$. The proof is then essentially the same as the proof of lemma 3.5.

The fundamental result is the analogue of theorem B of [10].

**Theorem 7.7** Let $\mathcal{W}$ be a compact suborbifold of $\mathcal{O}$. Let $S$ be a closed neighborhood in $\partial \mathcal{O}$ of $S \cap \partial \mathcal{W}$, and let $L$ be a neighborhood of $\mathcal{W}$ in $\mathcal{O}$. Then $\text{Imb}^L(\mathcal{W}, \mathcal{O} \text{ rel } S)$ admits local $\text{Diff}^L(\mathcal{O} \text{ rel } S)$ cross-sections.

**Proof of 7.7:** By proposition 3.2, it suffices to find a local cross-section at the inclusion $i_{\mathcal{W}}$. Choose a compact neighborhood $K$ of $\mathcal{W}$ with $K \subseteq L$. Using lemmas 7.6 and 7.5, there exist continuous maps $X: (\text{Imb}_H)^{<\delta}(\tilde{W}, \tilde{O}) \to (\mathcal{X}_H)^{<1/2}(\tilde{W}, T(\tilde{O}))$ and $k: (\mathcal{X}_H)^{<1/2}(\tilde{W}, T(\tilde{O})) \to \mathcal{X}_K^L(T(\tilde{O}))$. Let $J$ be a neighborhood as in lemma 7.4. On a sufficiently small neighborhood $\tilde{U}$ of $i_{\tilde{W}}$, the composition $\tilde{\chi} = FkX$ is defined and has image in $J$. Let $U$ be the imbeddings of $\mathcal{W}$ in $\mathcal{O}$ which admit a lift to $\tilde{U}$. By choosing $\tilde{U}$ small enough, we may ensure that the lift of an element of $U$ is unique. Define $\chi$ to be $\tilde{\chi}$ applied to the lift of an element of $U$ to $\tilde{U}$, followed by the projection of $\text{Diff}^L_H(\tilde{O})$ to $\text{Diff}^K(\mathcal{O})$. 
For elements in $U \cap \text{Imb}^K(W, O_{\text{rel} S})$, each lift to $\tilde{U}$ that is sufficiently close to $i_{\tilde{W}}$ must agree with $i_{\tilde{W}}$ on $\tilde{S}$. So $U$ may be chosen small enough so that if $j \in U$ then $\tilde{j} \in \text{Imb}(\tilde{W}, \tilde{O}_{\text{rel} \tilde{S}})$. Then, $X(\tilde{j}(x)) = Z(x)$ for all $x \in \tilde{S}$, so $k(X)(x) = Z(x)$ for all $x \in \tilde{S}$. It follows that $\chi(j) \in \text{Diff}(O_{\text{rel} S})$.

**Corollary 7.8** Let $W$ be a compact suborbifold of $O$, which is either properly imbedded or codimension-zero. Let $S$ be a closed neighborhood in $\partial O$ of $S \cap \partial W$, and let $L$ be a neighborhood of $W$ in $O$. Then the restriction $\text{Diff}^L(O_{\text{rel} S}) \to \text{Imb}^L(W, O_{\text{rel} S})$ is locally trivial.

**Corollary 7.9** Let $V$ and $W$ be suborbifolds of $O$, with $W \subset V$. Assume that $W$ compact, and is either properly imbedded or codimension-zero. Let $S$ be a closed neighborhood in $\partial O$ of $S \cap \partial W$, and let $L$ be a neighborhood of $W$ in $O$. Then the restriction $\text{Imb}^L(V, O_{\text{rel} S}) \to \text{Imb}^L(W, O_{\text{rel} S})$ is locally trivial.

**8 Singular fiberings**

We will say that a continuous surjection $p: \Sigma \to O$ of compact connected orbifolds is a *singular fibering* if there exists a commutative diagram

$$
\begin{array}{ccc}
\Sigma & \xrightarrow{\tilde{p}} & \tilde{O} \\
\downarrow{\sigma} & & \downarrow{\tau} \\
\Sigma & \xrightarrow{p} & O
\end{array}
$$

in which

(i) $\tilde{\Sigma}$ and $\tilde{O}$ are manifolds, and $\sigma$ and $\tau$ are regular orbifold coverings with groups of covering transformations $G$ and $H$ respectively,

(ii) $\tilde{p}$ is surjective and locally trivial, and

(iii) the fibers of $p$ and $\tilde{p}$ are path-connected.

The class of singular fiberings includes many Seifert fiberings, for example all compact 3-dimensional Seifert manifolds $\Sigma$ except the lens spaces with one or two exceptional orbits (see for example [13]). For some of those lens spaces,
\( \mathcal{O} \) fails to have an orbifold covering by a manifold. On the other hand, it is a much larger class than Seifert fiberings, because no structure as a homogeneous space is required on the fiber.

For mappings there is a complete analogy with the fibered case, where now \( \text{Diff}_f(\Sigma) \) is by definition the quotient of the group of fiber-preserving \( G \)-equivariant diffeomorphisms \( (\text{Diff}_G)_f(\Sigma) \) by its normal subgroup \( G \), and so on. A suborbifold \( W \) of \( \Sigma \) is called \textit{vertical} if it is a union of fibers. In this case the preimage \( \tilde{W} \) of \( W \) in \( \tilde{\Sigma} \) is a submanifold, and we can speak of \( \text{Imb}_f(W, \Sigma) \) and \( \text{Imb}_v(W, \Sigma) \).

Following our usual notations, we put \( \partial_v \Sigma = p^{-1}(\partial \mathcal{O}) \) and \( \partial_v W = W \cap \partial_v \Sigma \).

Since \( \mathcal{O} \) is compact, lemma 7.2 shows that a (complete) Riemannian metric on \( \tilde{\mathcal{O}} \) can be chosen so that \( H \) acts as isometries, and moreover so that the metric on \( \tilde{\mathcal{O}} \) is a product near the boundary. Next we will sketch how to obtain a \( G \)-equivariant metric which is a product near \( \partial_h \tilde{\Sigma} \) and near \( \partial_v \tilde{\Sigma} \). If \( \partial_v \tilde{\Sigma} \) is empty, we simply apply lemma 7.2. Assume that \( \partial_v \tilde{\Sigma} \) is nonempty. Construct a \( G \)-equivariant collar of \( \partial_h \tilde{\Sigma} \), and use it to obtain a \( G \)-equivariant metric such that the \( I \)-fibers of \( \partial_h \tilde{\Sigma} \times I \) are vertical. If \( \partial_v \tilde{\Sigma} \) is also nonempty, put \( Y = \partial_h \tilde{\Sigma} \cap \partial_v \tilde{\Sigma} \). We will follow the construction in the last paragraph of section 4. Denote the collar of \( \partial_h \tilde{\Sigma} \) by \( \partial_h \tilde{\Sigma} \times [0,2]_1 \). Assume that the metric on \( \partial_h \tilde{\Sigma} \) was a product on a collar \( Y \times [0,2]_2 \) of \( Y \) in \( \partial_h \tilde{\Sigma} \). Next, construct a \( G \)-equivariant collar \( \partial_v \tilde{\Sigma} \times [0,2]_2 \) of \( \partial_v \tilde{\Sigma} \) whose \( [0,2]_2 \)-fiber at each point of \( Y \times [0,2]_1 \) agrees with the \( [0,2]_2 \)-fiber of the collar of \( Y \) in \( \partial_\Sigma \times \{t\} \). Then, the product metric on \( \partial_v \tilde{\Sigma} \times [0,2]_2 \) agrees with the product metric of \( \partial_h \tilde{\Sigma} \times [0,2]_1 \), where they overlap, and the \( G \)-equivariant patching can be done to obtain a metric which is a product near \( \partial_v \tilde{\Sigma} \) without losing the property that it is a product near \( \partial_h \tilde{\Sigma} \). We will always assume that the metrics have been selected with these properties. In particular, \( G \) preserves the vertical and horizontal parts of vectors.

Some basic observations about singular fiberings will be needed.

**Lemma 8.1** The action of \( G \) preserves the fibers of \( \tilde{p} \). Moreover:

(i) If \( g \in G \), then there exists an element \( h \in H \) such that \( \tilde{p}g = h\tilde{p} \).

(ii) If \( h \in H \), then there exists an element \( g \) of \( G \) such that \( \tilde{p}g = h\tilde{p} \).

(iii) If \( x \in \Sigma \), then \( \tau^{-1} p(x) = \tilde{p}\sigma^{-1}(x) \).
§8. Singular fiberings

Proof of 8.1: Suppose that $\tilde{p}(x) = \tilde{p}(y)$. For $g \in G$, we have $\tau \tilde{p}(g(x)) = p\sigma(g(x)) = p\sigma(x) = \tau \tilde{p}(x) = \tau \tilde{p}(y) = \tau \tilde{p}(g(y))$. Since the fibers of $\tilde{p}$ are path-connected, and the fibers of $\tau$ are discrete, this implies that $g(x)$ and $g(y)$ lie in the same fiber of $\tilde{p}$. For (i), let $g \in G$. Since $g$ preserves the fibers of $\tilde{p}$, it induces a map $h$ on $\tilde{O}$. Given $x \in \tilde{O}$, choose $y \in \tilde{S}$ with $\tilde{p}(y) = x$. Then $\tau h(x) = \tau \tilde{p}(g(y)) = p\sigma(g(y)) = p\sigma(y) = \tau \tilde{p}(y) = \tau(x)$ so $h \in H$.

To prove (ii), suppose $h$ is any element of $H$. Let $\text{sing}(\mathcal{O})$ denote the singular set of $\mathcal{O}$. Choose $a \in \tilde{O} - \tau^{-1}(\text{sing}(\mathcal{O}))$, choose $s \in \tilde{S}$ with $\tilde{p}(s) = a$, and choose $s'' \in \tilde{S}$ with $\tilde{p}(s'') = h(a)$. Since $p\sigma(s) = \tau \tilde{p}(s) = \tau \tilde{p}(s'') = p\sigma(s'')$, $\sigma(s)$ and $\sigma(s'')$ must lie in the same fiber of $p$. Since the fiber is path-connected, there exists a path $\beta$ in that fiber from $\sigma(s'')$ to $\sigma(s)$. Let $\tilde{\beta}$ be its lift in $\tilde{S}$ starting at $s''$ and let $s'$ be the endpoint of this lift, so that $\sigma(s') = \sigma(s)$. Note that $\tilde{p}(s') = \tilde{p}(s'') = h(a)$ since $\tilde{\beta}$ lies in a fiber of $\tilde{p}$. Since $\sigma(s') = \sigma(s'')$, there exists a covering transformation $h \in G$ with $g(s') = h'$. To show that $\tilde{p}g = h\tilde{p}$, it is enough to verify that they agree on the dense set $\tilde{p}^{-1}(\tilde{O} - \tau^{-1}(\text{sing}(\mathcal{O})))$. Let $t \in \tilde{p}^{-1}(\tilde{O} - \tau^{-1}(\text{sing}(\mathcal{O})))$ and choose a path $\gamma$ in $\tilde{p}^{-1}(\tilde{O} - \tau^{-1}(\text{sing}(\mathcal{O})))$ from $s$ to $t$. Since $g \in G$, we have $p\sigma\gamma = p\sigma g\gamma$. Therefore $\tau \tilde{p}\gamma = \tau \tilde{p}g\gamma$, and so $\tilde{p}g\gamma$ is the unique lift of $p\sigma\gamma$ starting at $\tilde{p}g(s) = h(a)$. But this lift equals $h\tilde{p} \gamma$, so $h\tilde{p}(t) = \tilde{p}g(t)$.

For (iii), fix $z_0 \in \sigma^{-1}(x)$ and let $y_0 = \tilde{p}(z_0)$. Suppose $y \in \tilde{p}\sigma^{-1}(x)$. Choose $z \in \sigma^{-1}(x)$ with $\tilde{p}(z) = y$. Since $\sigma$ is a regular covering, there exists $g \in G$ such that $g(z) = z_0$. By (i), $g$ induces $h$ on $\tilde{O}$, and $h(y) = h\tilde{p}(z) = \tilde{p}g(z) = \tilde{p}(z_0) = y_0$. Therefore $\tau(y) = \tau(h(y)) = \tau(y_0) = \tau \tilde{p}(z_0) = p\sigma(z_0) = p(x)$ so $y \in \tau^{-1}(p(x))$. For the opposite inclusion, suppose that $y \in \tau^{-1}(p(x))$, so $\tau(y) = p(x) = \tau(y_0)$. Since $\sigma$ is regular, there exists $h \in H$ such that $h(y_0) = y$. Let $g$ be as in (ii). Then $y = h(y_0) = h\tilde{p}(z_0) = \tilde{p}g(z_0)$, and $\sigma(g(z_0)) = \sigma(z_0) = x$ so $y \in \tilde{p}(\sigma^{-1}(x))$.

One consequence of lemma 8.1 is that (smooth nonsingular) paths in $\tilde{O}$ have horizontal lifts in $\tilde{S}$. To see this, we first claim that the horizontal lifts of any vector $\omega$ in $T(\tilde{O})$ have bounded lengths. Fix a compact subset $C$ of $\tilde{S}$ such that $\sigma(C) = \Sigma$. Let $H_x \omega$ be the set of $H$-translates of $\omega$. Since $C$ is compact and $H_x \omega$ is closed, the lengths of the horizontal lifts of vectors in $H_x \omega$ to vectors in $T(\tilde{S})|_C$ are bounded by some $L$. If $\tilde{\omega}$ is any lift of $\omega$, there exists $g \in G$ such that $g \tilde{\omega} = \tilde{\omega}$ in $T(\tilde{S})|_C$. By lemma 8.1(i), there exists $h \in H$ such that $\tilde{p}_*(g_\omega) = h_\omega$. Since $g_\omega \tilde{\omega}$ is a horizontal lift of $h_\omega(\omega)$ and $g_\omega$ is an isometry,
\[ \|\tilde{\omega}\| = \|g_*\tilde{\omega}\| \leq L, \] proving the claim. Since the metric on \( \tilde{\Sigma} \) is complete, the claim shows that a path in \( \tilde{\Omega} \) could only fail to lift if a partial lift started in \( \tilde{\Sigma} - \partial_h \tilde{\Sigma} \) and then reached a point of \( \partial_h \tilde{\Sigma} \), impossible since the metric is a product near \( \partial_h \tilde{\Sigma} \).

Since horizontal lifts exist, the aligned exponential \( \text{Exp}_a \) of \( \tilde{\Sigma} \) is defined. Moreover, it is \( G \)-equivariant: since \( G \) consists of fiber-preserving isometries, \( \text{Exp}_a \) is \( G \)-equivariant, and since \( G \) preserves horizontal parts of vectors, it preserves horizontal lifts.

The notations \( \mathcal{A}(\tilde{W}, T(\tilde{\Sigma})) \) and \( \mathcal{A}(T(\tilde{\Sigma})) \) and the map \( F_a: \mathcal{A}(T(\tilde{\Sigma})) \rightarrow \text{Maps}(\tilde{\Sigma}, \tilde{\Sigma}) \) are analogous to those in section 5.

**Theorem 8.2** Let \( S \) be a closed subset of \( \partial \mathcal{O} \), and let \( T = p^{-1}(S) \). Then \( \text{Diff}(\mathcal{O} \text{ rel } S) \) admits local \( \text{Diff}_f(\Sigma \text{ rel } T) \) cross-sections.

**Proof of 8.2:** Lemma 7.6, with \( W = \mathcal{O} \), provides \( X: \tilde{U}_\delta \rightarrow (\mathcal{X}_H)_{<1/2}(T(\tilde{\Omega})) \), where \( \tilde{U}_\delta = \{ f \in \text{Diff}_H(\tilde{\Omega}) \mid d(f(x), x) < \delta \text{ for all } x \in \tilde{\Omega} \} \). Let \( h \in \text{Diff}(\mathcal{O}) \) and let \( \tilde{h} \in \text{Diff}_H(\tilde{\Omega}) \) be a lift of \( h \). For every \( \tilde{g} \in \tilde{U}_\delta \), \( \text{Exp}(X(\tilde{g}\tilde{h}^{-1}(x))) = \tilde{g}\tilde{h}^{-1}(x) \). Define \( \tilde{\chi}: \tilde{U}_\delta \rightarrow \mathcal{A}_G(T(\tilde{\Sigma})) \) by
\[ \tilde{\chi}(\tilde{g})(x) = (\tilde{p}_x|_{H_\delta})^{-1}(X(\tilde{g}\tilde{h}^{-1})(\tilde{p}(x))). \]

The boundary tangency conditions are clearly satisfied, and \( \text{Exp}_a(\tilde{\chi}(\tilde{g})(x)) \) exists since it is the horizontal lift of a geodesic from \( \tilde{p}(x) \) to \( \tilde{g}\tilde{h}^{-1}(\tilde{p}(x)) \). To see that \( \tilde{\chi}(\tilde{g}) \) is \( G \)-equivariant, suppose \( \gamma \in G \). By lemma 8.1(i), \( \gamma \) induces \( \lambda \in H \). We have
\[ \tilde{\chi}(\tilde{g})(\gamma(x)) = (\tilde{p}_x|_{H_\delta})^{-1}(X(\tilde{g}\tilde{h}^{-1})(\gamma(\tilde{p}(x)))) = (\tilde{p}_x|_{H_\delta})^{-1}(X(\tilde{g}\tilde{h}^{-1})(\lambda\tilde{p}(x))) = (\tilde{p}_x|_{H_\delta})^{-1}(\lambda_xX(\tilde{g}\tilde{h}^{-1})(\tilde{p}(x))) = \gamma_x(\tilde{p}_x|_{H_\delta})^{-1}(X(\tilde{g}\tilde{h}^{-1})(\tilde{p}(x))) = \gamma_x\tilde{\chi}(\tilde{g})(x), \]
the penultimate equality using the fact that \( G \) preserves the horizontal subspaces.

Let \( \tilde{U} = \tilde{\chi}^{-1}(J) \), where \( J \) is a neighborhood of \( 1_{\tilde{\Sigma}} \) as in lemma 7.4. Let \( U \) be a neighborhood of \( h \) consisting of elements having a lift in \( \tilde{U} \). Since \( G \) is a discrete subgroup of \( \text{Diff}_G(\tilde{\Sigma}) \), we may choose \( \delta \) small enough to ensure that
these lifts are unique. Now we can define \( \chi: U \to \text{Diff}_f(\Sigma) \) by putting \( \chi(g) \) equal to the diffeomorphism induced on \( \Sigma \) by \( F_a \tilde{\chi}(\tilde{g}) \).

From proposition 3.1, we have immediately

**Theorem 8.3** Let \( S \) be a closed subset of \( \partial \mathcal{O} \), and let \( T = p^{-1}(S) \). Then \( \text{Diff}_f(\Sigma \text{ rel } T) \to \text{Diff}(\mathcal{O} \text{ rel } S) \) is locally trivial.

We now extend lemmas 6.1 and 6.2 to the singular fibered case.

**Lemma 8.4** Let \( W \) be a vertical suborbifold of \( \Sigma \). Let \( T \) be a closed fibered neighborhood in \( \partial, \Sigma \) of \( T \cap \partial, W \). Then for all sufficiently small \( \delta \), there exists a continuous map \( k: (\mathcal{A}_G)_{<\delta}(\tilde{W}, T(\tilde{\Sigma})) \to \mathcal{A}_G(T(\tilde{\Sigma})) \) such that \( k(X)(x) = X(x) \) for all \( x \in \tilde{W} \) and \( X \in (\mathcal{A}_G)_{<\delta}(W, T(\tilde{\Sigma})) \). If \( X(x) = Z(x) \) for all \( x \in \tilde{T} \cap \partial, \tilde{W} \), then \( k(X)(x) = Z(x) \) for all \( x \in \tilde{T} \). Furthermore, \( k((\mathcal{V}_G)_{<\delta}(\tilde{W}, T(\tilde{\Sigma}))) \subset \mathcal{V}_G(T(\tilde{\Sigma})) \).

**Proof of 8.4:** As with lemma 3.4, the positive codimension and codimension-zero cases are similar, so we only discuss the former. Let \( W \) be the image of \( W \) in \( \mathcal{O} \), and denote \( \tau^{-1}W \) by \( \tilde{W} \). By lemma 8.1(iii), \( \mathcal{V} = \tilde{p}(\tilde{W}) \), and by lemma 8.1(ii), it is \( H \)-invariant. Since it is a submanifold of \( \mathcal{O} \), it follows that \( W \) is a suborbifold of \( \mathcal{O} \). A section \( X \in \mathcal{A}_G(\tilde{W}, T(\tilde{\Sigma})) \) induces a well-defined section \( \tilde{p}_*X \in \mathcal{X}(\tilde{W}, T(\tilde{\mathcal{O}})) \). By lemma 8.1(ii), \( \tilde{p}_*X \) is \( H \)-equivariant.

We claim that there exists a positive \( \delta \) so that if \( X \in (\mathcal{A}_G)_{<\delta}(\tilde{W}, T(\tilde{\Sigma})) \) then \( \tilde{p}_*X \in (\mathcal{X}_H)_{<1/2}(\tilde{W}, T(\tilde{\mathcal{O}})) \). For if not, there would be a sequence \( x_i \) in \( \tilde{W} \) such that \( \|X(x_i)\| \to 0 \) but \( \|\tilde{p}_*X(\tilde{p}(x_i))\| \geq 1/2 \). Since \( W \) is compact, there exists a compact subset \( C \subset \tilde{W} \) such that \( \sigma(C) = W \). There exist elements \( g_i \in G \) so that \( g_i(x_i) \in C \), and if \( h_i \in H \) are obtained using lemma 8.1(i) then \( \|X(g_i(x_i))\| = \|X(x_i)\| \) while \( \|\tilde{p}_*X(\tilde{p}(g_i)(x_i))\| = \|\tilde{p}_*X(h_i\tilde{p}(x_i))\| = \|\tilde{p}_*X(\tilde{p}(x_i))\| \geq 1/2 \). So we may assume that the \( x_i \) lie in \( C \), hence that they converge to \( x \in C \). Then, \( \|X(x)\| = 0 \) but \( \|\tilde{p}_*X(\tilde{p}(x))\| \geq 1/2 \), a contradiction.

We now follow the proof of lemma 6.1. Let \( k_{\tilde{G}}: (\mathcal{X}_H)_{<1/2}(\tilde{W}, T(\tilde{\mathcal{O}})) \to \mathcal{X}_H(T(\tilde{\mathcal{O}})) \) be obtained using lemma 7.5. Let \( \nu_\varepsilon(\tilde{W}) \) be the \( \varepsilon \)-normal bundle of \( \tilde{W} \). Since \( W \) is compact, for sufficiently small \( \varepsilon \), \( j_a: \nu_\varepsilon(\tilde{W}) \to \Sigma \) defined by \( j_a(\omega) = \text{Exp}_a(\omega) \) and carries \( \nu_\varepsilon(\tilde{W}) \) diffeomorphically to a neighborhood
§8. Singular fiberings

of \( \widetilde{W} \) in \( \widetilde{\Sigma} \). Since \( W \) is compact, we may choose \( \epsilon \) small enough so that \( j_\alpha(\omega) \in \partial h \widetilde{\Sigma} \times I \) only when \( \pi(\omega) \in \partial h \widetilde{\Sigma} \times I \).

Since \( G \) acts as isometries and preserves horizontal lifts, the aligned parallel translation \( P_\alpha \) is \( G \)-equivariant. Using lemma 7.1 there exists a smooth \( G \)-equivariant function \( \alpha: \widetilde{\Sigma} \to [0, 1] \) which is identically 1 on \( \widetilde{W} \) and identically 0 on \( \widetilde{\Sigma} - j(\nu/2(\widetilde{W})) \). Define \( k: \mathcal{A}_G(\mathcal{W}, T(\widetilde{\Sigma})) \) by

\[
k(X)(x) = (\tilde{p}_x|_{H_x})^{-1}(k_{\tilde{\Theta}}(p_x X)(p(x))) + k_{\tilde{\Sigma}}(X_v)(x).
\]

Lemma 8.5 Let \( W \) be a vertical suborbifold of \( \Sigma \). For small \( \delta > 0 \), there exists a continuous map

\[
X: ((\text{Imb}_G)_f)_<\delta(\widetilde{W}, \widetilde{\Sigma}) \to \mathcal{A}_G(\widetilde{W}, T(\widetilde{\Sigma}))
\]

such that \( \exp_a(X(j)(x)) = j(x) \) for all \( x \in \widetilde{W} \) and \( j \in ((\text{Imb}_G)_f)_<\delta(\widetilde{W}, \widetilde{\Sigma}) \). Moreover, \( X(((\text{Imb}_G)_v)_<\delta(\widetilde{W}, \widetilde{\Sigma})) \subseteq \mathcal{V}_G(\widetilde{W}, T(\widetilde{\Sigma})) \), and if \( j(x) = i_{\widetilde{W}}(x) \) then \( X(j)(x) = Z(x) \).

Proof of 8.5: Let \( N_\epsilon(\widetilde{W}) \) be as defined before the proof of lemma 6.2. Since \( W \) is compact, we can choose \( \epsilon \) small enough to ensure the local diffeomorphism condition. Choose \( \delta \) small enough so that \( j(x) \in \exp_a(N_\epsilon(\widetilde{W}) \cap T_x(\widetilde{\Sigma})) \) for every \( x \in \widetilde{W} \) and \( j \in ((\text{Imb}_G)_f)_<\delta(\widetilde{W}, \widetilde{\Sigma}) \). Define \( X(j)(x) \) to be the unique vector in \( N_\delta(\widetilde{W}) \cap T_x(\widetilde{\Sigma}) \) such that \( \exp_a(X(j)(x)) = j(x) \).

Theorem 8.6 Let \( W \) be a vertical suborbifold of \( \Sigma \). Let \( T \) be a closed fibered neighborhood in \( \partial_v \Sigma \) of \( T \cap \partial_v W \). Then
§8. Singular fiberings

(i) $\text{Imb}_f(W, \Sigma \text{ rel } T)$ admits local $\text{Diff}_f(\Sigma \text{ rel } T)$ cross-sections, and

(ii) $\text{Imb}_v(W, \Sigma \text{ rel } T)$ admits local $\text{Diff}_v(\Sigma \text{ rel } T)$ cross-sections.

**Proof of 8.6:** By proposition 3.2, it suffices to construct local cross-sections at the inclusion $i_W$. Obtain $k: (A_G)_{<\delta}(\tilde{W}, T(\tilde{\Sigma})) \to (A_G)_{<1/2}(T(\tilde{\Sigma}))$, and $X: ((\text{Imb}_G)_f)_{<\delta}(\tilde{W}, \tilde{\Sigma}) \to A_G(\tilde{W}, T(\tilde{\Sigma}))$ using lemmas 8.4 and 8.5. Fix a neighborhood $\tilde{U}$ of $i_\tilde{W}$ small enough so that $X(\tilde{U}) \subseteq (A_G)_{<\delta}(\tilde{W}, T(\tilde{\Sigma}))$. Let $U$ be a neighborhood of $i$ small enough so that each element $j$ of $U$ has a unique lift $\tilde{j}$ into $\tilde{U}$, and so that if $j$ agrees with $i_W$ on $\partial_v W$ then $\tilde{j}$ agrees with $i_{\tilde{W}}$ on $\partial_v \tilde{W}$. For $j \in U$, define $\chi(j)$ to be the element of $\text{Diff}_f(\Sigma \text{ rel } T)$ induced by $F_a k X(\tilde{j})$.

8.6

As in section 6, we have the following immediate corollaries.

**Corollary 8.7** Let $W$ be a vertical suborbifold of $\Sigma$. Let $T$ be a fibered neighborhood in $\partial_v \Sigma$ of $T \cap \partial_v W$. Then the following restrictions are locally trivial:

(i) $\text{Diff}_f(\Sigma \text{ rel } T) \to \text{Imb}_f(W, \Sigma \text{ rel } T)$, and

(ii) $\text{Diff}_v(\Sigma \text{ rel } T) \to \text{Imb}_v(W, \Sigma \text{ rel } T)$.

**Corollary 8.8** Let $V$ and $W$ be vertical suborbifolds of $\Sigma$, with $W \subseteq V$. Let $T$ be a closed fibered neighborhood in $\partial_v \Sigma$ of $T \cap \partial_v W$. Then the following restrictions are locally trivial:

(i) $\text{Imb}_f(V, \Sigma \text{ rel } T) \to \text{Imb}_f(V, \Sigma \text{ rel } T)$, and

(ii) $\text{Imb}_v(V, \Sigma \text{ rel } T) \to \text{Imb}_v(W, \Sigma \text{ rel } W)$.

**Theorem 8.9** Let $W$ be a vertical suborbifold of $\Sigma$. Let $T$ be a closed fibered neighborhood in $\partial_v \Sigma$ of $T \cap \partial_v W$, and let $S = p(T)$. Then all four maps in the following square are locally trivial:

$$
\begin{array}{ccc}
\text{Diff}_f(\Sigma \text{ rel } T) & \longrightarrow & \text{Imb}_f(W, \Sigma \text{ rel } T) \\
\downarrow & & \downarrow \\
\text{Diff}(\mathcal{O} \text{ rel } S) & \longrightarrow & \text{Imb}(p(W), \mathcal{O} \text{ rel } S)
\end{array}
$$
9 Restricting to the boundary or the basepoint

Our restriction theorems deal with the case when the suborbifold is properly imbedded. By a simple doubling trick, we can also extend to restriction to suborbifolds of the boundary.

Proposition 9.1 Let $\Sigma \rightarrow O$ be a singular fibering. Let $S$ be a suborbifold of $\partial O$, and let $T = p^{-1}(S)$. Then

(a) $\text{Imb}(S, \partial O)$ admits local $\text{Diff}(O)$ cross-sections.

(b) $\text{Imb}_f(T, \partial_v \Sigma)$ admits local $\text{Diff}_f(\Sigma)$ cross-sections.

Proof of 9.1: For (a), we first show that $\text{Diff}(\partial O)$ admits local $\text{Diff}(O)$ cross-sections. Let $\Delta$ be the double of $O$ along $\partial O$, and regard $O$ as a suborbifold of $\Delta$ by identifying it with one of the two copies of $O$ in $\Delta$. By theorem 7.7, $\text{Imb}(\partial O, \Delta)$ admits local $\text{Diff}(\Delta)$ cross-sections. We may regard $\text{Diff}(\partial O)$ as a subspace of $\text{Imb}(\partial O, \Delta)$. Suppose $\chi: U \rightarrow \text{Diff}(\Delta)$ is a local cross-section at a point in $\text{Imb}(\partial O, \Delta)$ that lies in $\text{Diff}(\partial O)$. By composing with the diffeomorphism of $\Delta$ that interchanges the two copies of $O$, and reducing the size of $U$ if necessary, we may assume that $\chi$ carries the elements of $U$ that preserve $\partial O$ to diffeomorphisms that preserve $O$. Then a local $\text{Diff}(O)$ cross-section on $U \cap \text{Diff}(\partial O)$ is defined by sending $g$ to $\chi(g)|_O$.

The proof of (b) is similar. Double $\Sigma$ along $\partial_v \Sigma$ and apply theorem 8.6, to produce local $\text{Diff}_f(\Sigma)$ cross-sections for $\text{Diff}_f(\partial_v \Sigma)$. Apply it again to produce local $\text{Diff}_f(\partial_v \Sigma)$ cross-sections for $\text{Imb}_f(T, \partial_v \Sigma)$. Their composition, where defined, is a local $\text{Diff}_f(\Sigma)$ cross-section for $\text{Imb}_f(T, \partial_v \Sigma)$.

9.1 An immediate consequence is
Corollary 9.2 Let $\Sigma \to O$ be a singular fibering. Let $S$ be a suborbifold of $\partial O$, and let $T = p^{-1}(S)$. Then $\text{Diff}(O) \to \text{Imb}(S, \partial O)$ and $\text{Diff}_f(\Sigma) \to \text{Imb}_f(T, \partial_v \Sigma)$ are locally trivial. In particular, $\text{Diff}(O) \to \text{Diff}(\partial O)$ and $\text{Diff}_f(\Sigma) \to \text{Diff}_f(\partial_v \Sigma)$ are locally trivial.

Here are two other consequences which are applied in [9].

Corollary 9.3 Let $W$ be a suborbifold of $O$. Then $\text{Imb}(W, O) \to \text{Imb}(W \cap \partial O, \partial O)$ is locally trivial.

Proof of 9.3: By theorem 7.7, $\text{Imb}(W \cap \partial O, \partial O)$ admits local $\text{Diff}(\partial O)$ cross-sections, and by proposition 9.1, $\text{Diff}(\partial O)$ admits local $\text{Diff}(O)$ cross-sections. Composing them gives local $\text{Diff}(O)$ cross-sections for $\text{Imb}(W \cap \partial O, \partial O)$.

Corollary 9.4 Let $W$ be a vertical suborbifold of $\Sigma$. Then $\text{Imb}_f(W, \Sigma) \to \text{Imb}_f(W \cap \partial_v \Sigma, \partial_v \Sigma)$ is locally trivial.

Proof of 9.4: Theorem 8.6, applied to $\partial_v \Sigma$, and proposition 9.1 show that $\text{Imb}_f(W \cap \partial_v \Sigma, \partial_v \Sigma)$ admits local $\text{Diff}_f(\Sigma)$ cross-sections.

Many applications of the fibration $\text{Diff}(M) \to \text{Imb}(V, M)$ concern the case when the submanifold is a single point. Since in the fibered case a single point is not usually a vertical submanifold, this case is not directly covered by our previous theorems. The next proposition allows nonvertical suborbifolds that are contained in a single fiber, so applies when the submanifold is a single point. To set notation, let $p: \Sigma \to O$ be a singular fibering. Let $P$ be a suborbifold of $\Sigma$ which is contained in a single fiber $F$. Let $T$ be a fibered closed subset of $\partial_v \Sigma$. By $\text{Imb}_t(P, \Sigma \text{ rel } T)$ we denote the orbifold imbeddings whose image is contained in a single fiber of $\Sigma$, which restrict to the identity on $P \cap T$, and which map $P \cap (\partial \Sigma - T)$ into $\partial \Sigma - T$.

Proposition 9.5 Let $T$ be a fibered closed subset of $\partial_v \Sigma$, which is a neighborhood in $\partial_v \Sigma$ of $P \cap T$. Then $\text{Imb}_t(P, \Sigma \text{ rel } T)$ admits local $\text{Diff}_f(\Sigma \text{ rel } T)$ cross-sections.
Proof of 9.5: Notice that $p(P)$ is a point and is a properly imbedded sub-orbifold of $O$, with orbifold structure determined by the local group at $p(P)$. Each imbedding $i \in \text{Imb}_{t}(P, \Sigma)$ induces an orbifold imbedding $pi: p(P) \to O$. Let $S=p(T)$.

By proposition 3.2, it suffices to produce a local cross-section at the inclusion $i_P$. By theorem 7.7, $\text{Imb}(p(P), O \text{ rel } S)$ has local $\text{Diff}(O \text{ rel } S)$ cross-sections, and by proposition 8.2, $\text{Diff}(O \text{ rel } S)$ has local $\text{Diff}f(\Sigma \text{ rel } T)$ cross-sections. A suitable composition of these gives a local $\text{Diff}f(\Sigma \text{ rel } T)$ cross-section $\chi_1$ for $\text{Imb}(p(P), O \text{ rel } S)$ at $pi_P$. As remarked in section 3, we may assume that $\chi_1(p_iP)$ is the identity diffeomorphism of $\Sigma$. By corollary 7.8, there exists a local $\text{Diff}(F \text{ rel } T \cap F)$ cross-section $\chi_2$ for $\text{Imb}(P, F \text{ rel } T \cap F)$ at $i_P$, and we may assume that $\chi_2(i_P)$ is the identity diffeomorphism of $F$. Let $\chi_3$ be a local $\text{Diff}f(\Sigma \text{ rel } T)$ cross-section for $\text{Imb}_f(F, \Sigma \text{ rel } T)$ at $i_F$ given by corollary 8.7. Regarding $\text{Diff}(F \text{ rel } F \cap T)$ as a subspace of $\text{Imb}_f(F, \Sigma \text{ rel } T)$, we may assume that the composition $\chi_3\chi_2$ is defined. On a sufficiently small neighborhood of $i_P$ in $\text{Imb}_t(P, \Sigma \text{ rel } T)$ define $\chi(j) \in \text{Diff}f(\Sigma \text{ rel } T)$ by

$$\chi(j) = \chi_1(p(j)) (\chi_3\chi_2)(\chi_1(p(j))^{-1} \circ j) .$$

Then for $x \in P$ we have

$$\chi(j)i_P(x) = \chi_1(p(j)) (\chi_3\chi_2)(\chi_1(p(j))^{-1} \circ j)(x)$$
$$= \chi_1(p(j)) \chi_1(p(j))^{-1}j(x)$$
$$= j(x)$$

This yields immediately

**Corollary 9.6** Let $W$ be a vertical suborbifold of $\Sigma$ containing $P$. Then $\text{Diff}f(\Sigma \text{ rel } T) \to \text{Imb}_t(P, \Sigma \text{ rel } T)$ and $\text{Imb}_f(W, \Sigma \text{ rel } T) \to \text{Imb}_t(P, \Sigma \text{ rel } T)$ are locally trivial.

10 The space of Seifert fiberings of a Haken 3-manifold

Let $p: \Sigma \to O$ be a Seifert fibering of a Haken manifold $\Sigma$. As noted in section 8, $p$ is a singular fibering. Denote by $\text{diff}_f(\Sigma)$ the connected component of the
identity in $\text{Diff}_f(\Sigma)$, and similarly for other spaces of diffeomorphisms and imbeddings. The main result of this section is the following.

**Theorem 10.1** Suppose that $\Sigma$ is a Haken 3-manifold. Then the inclusion $\text{diff}_f(\Sigma) \to \text{diff}(\Sigma)$ is a weak homotopy equivalence.

Before proving theorem 10.1, we give an application. Each element of $\text{Diff}(\Sigma)$ carries the given fibering to an isomorphic fibering, and $\text{Diff}_f(\Sigma)$ is precisely the stabilizer of the given fibering under this action. Therefore it is reasonable to define the space of Seifert fiberings isomorphic to the given fibering to be the space of cosets $\text{Diff}(\Sigma)/\text{Diff}_f(\Sigma)$. Since $\text{Diff}_f(\Sigma)$ is a closed subgroup, the quotient $\text{Diff}(\Sigma) \to \text{Diff}(\Sigma)/\text{Diff}_f(\Sigma)$ is a principal fibering with fiber $\text{Diff}_f(\Sigma)$. As an immediate corollary to theorem 10.1, we will obtain:

**Theorem 10.2** Suppose that $\Sigma$ is a Haken 3-manifold. Then each path component of the space of Seifert fiberings of $\Sigma$ is weakly contractible.

**Proof of 10.2:** As sketched on p. 85 of [17], two fiber-preserving diffeomorphisms of $\Sigma$ that are isotopic are isotopic through fiber-preserving diffeomorphisms. This implies that $\pi_0(\text{Diff}_f(\Sigma)) \to \pi_0(\text{Diff}(\Sigma))$ is injective. By theorem 10.1, $\pi_q(\text{Diff}_f(\Sigma)) \to \pi_q(\text{Diff}(\Sigma))$ is an isomorphism for all $q \geq 1$. The theorem now follows from the homotopy exact sequence for the fibration $\text{Diff}(\Sigma) \to \text{Diff}(\Sigma)/\text{Diff}_f(\Sigma)$.

For compact Seifert fibered 3-manifolds, apart from a small list of well-known exceptions, every diffeomorphism is isotopic to a fiber-preserving diffeomorphism. So the following immediate corollary applies to most cases.

**Corollary 10.3** Suppose that $\Sigma$ is a Haken 3-manifold such that every diffeomorphism is isotopic to a fiber-preserving diffeomorphism. Then the space of Seifert fiberings of $\Sigma$ is weakly contractible.

The proof of theorem 10.1 will use the following lemma.

**Lemma 10.4** Let $\Sigma$ be a Haken Seifert fibered 3-manifold, and let $C$ be a fiber of $\Sigma$. Then each component of $\text{Diff}_v(\Sigma \text{ rel } C)$ is contractible.
Proof of 10.4: Since $\Sigma$ is Haken, the base orbifold of $\Sigma - C$ has nonpositive Euler characteristic and is not closed. It follows (see [13]) that $\Sigma - C$ admits an $\mathbb{H}^2 \times \mathbb{R}$ geometry. Thus there is an action of $\pi_1(\Sigma - C)$ on $\mathbb{H}^2 \times \mathbb{R}$ such that every element preserves the $\mathbb{R}$-fibers and acts as an isometry on the $\mathbb{H}^2$ factor. Let $B$ be the orbit space of $\Sigma - C$.

It suffices to show that $\text{diff}_v(\Sigma \text{ rel } C)$ is contractible. Let $N$ be a fibered solid torus neighborhood of $C$ in $\Sigma$. It is not difficult to see that $\text{diff}_v(\Sigma \text{ rel } C)$ deformation retracts to $\text{diff}_v(\Sigma \text{ rel } N)$, which can be identified with $\text{diff}_v(\Sigma - C \text{ rel } N - C)$, so it suffices to show that the latter is contractible. For $f \in \text{diff}_v(\Sigma - C \text{ rel } N - C)$, let $F$ be a lift of $f$ to $\mathbb{H}^2 \times \mathbb{R}$ that has the form $F(x, s) = (x, s + F_2(x, s))$, where $F_2(x, s) \in \mathbb{R}$. Since $f$ is vertically isotopic to the identity relative to $N - C$, we may moreover choose $F$ so that $F_2(x, s) = 0$ if $(x, s)$ projects to $N - C$. To see this, we choose the lift $F$ to fix a point in the preimage $W$ of $N - C$. Since $f$ is homotopic to the identity relative to $N - C$, $F$ is equivariantly homotopic to a covering translation relative to $W$. That covering translation fixes the point in $W$, and therefore must be the identity. Thus $F$ fixes $W$ and commutes with every covering translation.

Define $K_t$ by $K_t(x, s) = (x, s + (1 - t)F_2(x, s))$. Since $K_0 = F$ and $K_1$ is the identity, and each $K_t$ is the identity on the preimage of $N - C$, this will define a contraction of $\text{Diff}_v(\Sigma - C \text{ rel } N - C)$ once we have shown that each $K_t$ is equivariant. Let $\gamma \in \pi_1(\Sigma - C)$. From [13], $\text{Isom}(\mathbb{H}^2 \times \mathbb{R}) = \text{Isom}(\mathbb{H}^2) \times \text{Isom}(\mathbb{R})$, so we can write $\gamma(x, s) = (\gamma_1(x), \epsilon\gamma_2 + \gamma_2)$, where $\epsilon_\gamma = \pm 1$ and $\gamma_2 \in \mathbb{R}$. Since $F\gamma = \gamma F$, a straightforward calculation shows that

$$F_2(\gamma_1(x), \epsilon\gamma_2 + \gamma_2) = \epsilon\gamma_2 F_2(x, s).$$

Now we calculate

$$K_t\gamma(x, s) = K_t(\gamma_1(x), \epsilon\gamma_2 + \gamma_2) = (\gamma_1(x), \epsilon\gamma_2 + \gamma_2 + (1 - t)F_2(\gamma_1(x), \epsilon\gamma_2 + \gamma_2)) = (\gamma_1(x), \epsilon\gamma_2 + \gamma_2 + (1 - t)\epsilon\gamma_2 F_2(x, s)) = (\gamma_1(x), \epsilon\gamma_2 + (1 - t)F_2(x, s)) + \gamma_2 = \gamma(x, s + (1 - t)F_2(x, s)) = \gamma K_t(x, s)$$

showing that $K_t$ is equivariant.
Proof of 10.1: We first examine $\text{diff}_v(\Sigma)$. Choose a regular fiber $C$ and consider the restriction $\text{diff}_v(\Sigma) \to \text{imb}_v(C, \Sigma) \cong \text{diff}(C) \cong \text{diff}(S^1) \cong \text{SO}(2)$. By corollary 8.7(ii), this is a fibration. By lemma 10.4, each component of the fiber $\text{Diff}_v(\Sigma \text{ rel } C) \cap \text{diff}_v(\Sigma)$ is contractible. It follows by the exact sequence for this fibration that $\pi_q(\text{diff}_v(\Sigma)) \cong \pi_q(\text{SO}(2)) = 0$ for $q \geq 2$, and for $q = 1$ we have an exact sequence

$$0 \longrightarrow \pi_1(\text{diff}_v(\Sigma)) \longrightarrow \pi_1(\text{diff}(C)) \longrightarrow \pi_0(\text{Diff}(\Sigma \text{ rel } C) \cap \text{diff}_v(\Sigma)) \longrightarrow 0.$$  

We will first show that exactly one of the following holds.

a) $C$ is central and $\pi_1(\text{diff}_v(\Sigma)) \cong \mathbb{Z}$ generated by the vertical $S^1$-action.  

b) $C$ is not central and $\pi_1(\text{diff}_v(\Sigma))$ is trivial.

Suppose first that the fiber $C$ is central in $\pi_1(\Sigma)$. Then there is a vertical $S^1$-action on $\Sigma$ which moves the basepoint (in $C$) once around $C$. This maps onto the generator of $\pi_1(\text{diff}(C))$, so $\pi_1(\text{diff}_v(\Sigma)) \to \pi_1(\text{diff}(C))$ is an isomorphism. Therefore $\pi_1(\text{diff}_v(\Sigma))$ is infinite cyclic, with generator represented by the vertical $S^1$-action.

If the fiber is not central, then $\pi_1(\text{diff}(C)) \to \pi_0(\text{Diff}(\Sigma \text{ rel } C) \cap \text{diff}_v(\Sigma))$ carries the generator to a diffeomorphism of $\Sigma$ which induces an inner automorphism of infinite order on $\pi_1(\Sigma, x_0)$, where $x_0$ is a basepoint in $C$. Since elements of $\text{Diff}(\Sigma \text{ rel } C)$ fix the basepoint, this diffeomorphism (and its powers) are not in $\text{diff}(\Sigma \text{ rel } C)$. Therefore $\pi_1(\text{diff}(C)) \to \pi_0(\text{Diff}(\Sigma \text{ rel } C) \cap \text{diff}_v(\Sigma))$ is injective, so $\pi_1(\text{diff}_v(\Sigma))$ is trivial.

Now consider the fibration of theorem 8.3:

$$(\ast) \quad \text{Diff}_v(\Sigma) \cap \text{diff}_f(\Sigma) \longrightarrow \text{diff}_f(\Sigma) \longrightarrow \text{diff}(\mathcal{O}).$$

Observe that $\text{diff}(\mathcal{O})$ is homotopy equivalent to the identity component of the space of diffeomorphisms of the 2-manifold $\mathcal{O} - \mathcal{E}$, where $\mathcal{E}$ is the exceptional set. Since $\Sigma$ is Haken, this 2-manifold is either a torus, annulus, disc with one puncture, Mobius band, or Klein bottle, or a surface of negative Euler characteristic. Therefore $\text{diff}(\mathcal{O})$ is contractible unless $\chi(\mathcal{O}) = 0$, in which case its higher homotopy groups are all trivial, and its fundamental group is isomorphic to the center of $\pi_1(\mathcal{O})$. In the latter cases, the elements of $\pi_1(\mathcal{O})$
are classified by their traces at a basepoint of \( \mathcal{O} - \mathcal{E} \). From the exact sequence for the fibration \((*)\), it follows that \( \pi_q(\text{diff}_f(\Sigma)) = 0 \) for \( q \geq 2 \).

To complete the proof, we recall the result of Hatcher [2]: for \( M \) Haken, \( \pi_q(\text{diff}(M)) = 0 \) for \( q \geq 2 \) and is isomorphic to the center of \( \pi_1(M) \) for \( q = 1 \), and the elements of \( \pi_1(\text{diff}(M)) \) are classified by their traces at the basepoint. We already have \( \pi_q(\text{diff}_f(\Sigma)) = 0 \) for \( q \geq 2 \), so it remains to show that \( \pi_1(\text{diff}_f(\Sigma)) \to \pi_1(\text{diff}(\Sigma)) \) is an isomorphism.

**Case I:** \( \pi_1(\mathcal{O}) \) is centerless.

In this case \( \text{diff}(\mathcal{O}) \) is contractible, and either \( C \) generates the center or \( \pi_1(\Sigma) \) is centerless. The exact sequence associated to the fibration \((*)\) shows that \( \pi_1(\text{diff}_v(\Sigma)) \to \pi_1(\text{diff}_f(\Sigma)) \) is an isomorphism. Suppose \( C \) generates the center. Since \( \pi_1(\text{diff}_v(\Sigma)) \) is infinite cyclic generated by the vertical \( S^1 \)-action, Hatcher’s theorem shows that the composition \( \pi_1(\text{diff}_v(\Sigma)) \to \pi_1(\text{diff}_f(\Sigma)) \to \pi_1(\text{diff}(\Sigma)) \) is an isomorphism. Therefore \( \pi_1(\text{diff}_f(\Sigma)) \to \pi_1(\text{diff}(\Sigma)) \) is an isomorphism.

If \( \pi_1(\Sigma) \) is centerless, then \( \pi_1(\text{diff}(\Sigma)) = 0 \), \( \pi_1(\text{diff}_v(\Sigma)) \cong \pi_1(\text{diff}_f(\Sigma)) = 0 \), and again \( \pi_1(\text{diff}_f(\Sigma)) \to \pi_1(\text{diff}(\Sigma)) \) is an isomorphism.

**Case II:** \( \pi_1(\mathcal{O}) \) has center.

Assume first that \( \mathcal{O} \) is a torus. If \( \Sigma \) is the 3-torus, then by considering the exact sequence for the fibration \((*)\), one can check directly that the homomorphism \( \partial: \pi_1(\text{diff}(\mathcal{O})) \to \pi_0(\text{Diff}_v(\Sigma) \cap \text{diff}_f(\Sigma)) \) is the zero map. We obtain the exact sequence

\[
0 \to \mathbb{Z} \to \pi_1(\text{diff}_f(\Sigma)) \to \mathbb{Z} \times \mathbb{Z} \to 0.
\]

Since \( \text{diff}_f(\Sigma) \) is a topological group, \( \pi_1(\text{diff}_f(\Sigma)) \) is abelian and hence isomorphic to \( \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \). The traces of the generating elements generate the center of \( \pi_1(\Sigma) \), which shows that \( \pi_1(\text{diff}_f(\Sigma)) \to \pi_1(\text{diff}(\Sigma)) \) is an isomorphism.

Suppose that \( \Sigma \) is not a 3-torus. Then \( \Sigma = \mathcal{O} \times I/(x, 0) \cong (\phi(x), 1) \) for a homeomorphism \( \phi: \mathcal{O} \to \mathcal{O} \), \( \pi_1(\Sigma) = \langle a, b, t \mid tat^{-1} = a, [a, b] = 1, tbt^{-1} = a^n b \rangle \) for some integer \( n \), and the fiber \( a \) generates the center of \( \pi_1(\Sigma) \).

Let \( b_0 \) and \( t_0 \) be the image of the generators of \( b \) and \( t \) respectively in \( \pi_1(\mathcal{O}) \). Now \( \pi_1(\text{diff}(\mathcal{O})) \cong \mathbb{Z} \times \mathbb{Z} \) generated by elements whose traces represent the elements \( b_0 \) and \( t_0 \). By lifting these isotopies we see that \( \partial: \pi_1(\text{diff}(\mathcal{O})) \to \pi_0(\text{diff}_v(\Sigma)) \) is injective. Therefore \( \pi_1(\text{diff}_v(\Sigma)) \) is isomorphic to \( \pi_1(\text{diff}_f(\Sigma)) \), and the result follows as in case I.
Assume now that $O$ is a Klein bottle. As in the torus case we may view
$\Sigma = O \times I / (x, 0) \simeq (\phi(x), 1)$, $\pi_1(\Sigma) = \langle a, b, t \mid tat^{-1} = a^{-1}, [a, b] = 1, tbt^{-1} = a^{-n}b^{-1} \rangle$ for some integer $n$, with fiber $a$, and $\pi_1(O) = \langle b_0, t_0 \mid t_0b_0t_0^{-1} = b_0^{-1} \rangle$.

Now $\pi_1(\text{diff}(O))$ is generated by an isotopy whose trace represents the generator of the center of $\pi_1(\text{diff}(O))$, the element $t_0^2$. Observe that $\pi_1(\Sigma)$ has center if and only if $n = 0$. If $n = 0$, then it follows that $\partial: \pi_1(\text{diff}(O)) \to \pi_0(\text{Diff}_v(\Sigma) \cap \text{diff}_f(\Sigma))$ is the zero map. Hence $\pi_1(\text{diff}_f(\Sigma)) \to \pi_1(\text{diff}(O))$ is an isomorphism and the generator of $\pi_1(\text{diff}_f(\Sigma))$ is represented by an isotopy whose trace represents the element $t^2$. By Hatcher’s result, $\pi_1(\text{diff}_f(\Sigma)) \to \pi_1(\text{diff}(\Sigma))$ is an isomorphism. If $n \neq 0$, then $\partial: \pi_1(\text{diff}(O)) \to \pi_0(\text{Diff}_v(\Sigma) \cap \text{diff}_f(\Sigma))$ is injective. Since $\pi_1(\Sigma)$ is centerless, $\pi_1(\text{Diff}_v(\Sigma) \cap \text{diff}_f(\Sigma)) = 0$. This implies that $\pi_1(\text{diff}_f(\Sigma)) = 0$, and again Hatcher’s result applies.

The cases where $O$ is an annulus, disc with one puncture, or a Mobius band are similar to those of the torus and Klein bottle.

References


