Formula Sheet

$f(t)$	$\mathcal{L}(f(t))$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^a (a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$

The above Laplace transforms are valid for $s > 0$, with the exception of the Laplace transforms of $\sinh(kt)$ and $\cosh(kt)$ which are valid for $s > |k|$ and the Laplace transform of e^{at} which is valid for $s > a$.

x	$\Gamma(x)$
$n+1$ (for integer $n \geq 0$)	$n!$
$1/2$	$\sqrt{\pi}$
$3/2$	$\frac{1}{2}\sqrt{\pi}$
$5/2$	$\frac{3}{4}\sqrt{\pi}$
$7/2$	$\frac{15}{8}\sqrt{\pi}$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt$$

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$$

$$\mathcal{L}(f^{(n)}(t)) = s^n \mathcal{L}(f(t)) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$

$$\mathcal{L}\left(\int_0^t f(\theta)d\theta\right) = \frac{\mathcal{L}(f(t))}{s}$$

$$F(s) = \mathcal{L}(f(t)) \implies F(s-a) = \mathcal{L}(e^{at}f(t))$$

$$f(t) * g(t) = \int_0^t f(\theta)g(t-\theta)d\theta$$

$$\mathcal{L}(-tf(t)) = \frac{dF(s)}{ds}$$

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\sigma)d\sigma$$

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s), (\text{ valid for } s > c+a \text{ if } \mathcal{L}(f(t)) = F(s) \text{ is valid for } s > c)$$

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MATH 3113-170 Test III

Dr. Darren Ong

June 22, 2015 1:00pm-2:15pm

Answer the questions in the spaces provided on the question sheets. No calculators allowed.

Name: ____________________

1. (5 points) Write down $\mathcal{L}(f(t))$ for the following functions $f(t)$ (no work needs to be shown for this problem, and no partial credit will be given).

   (a) $f(t) = 0$

   (b) $f(t) = u(t-3)$

   (c) $f(t) = \sqrt[3]{t}$

   (d) $f(t) = e^{-2t}$

   (e) $f(t) = 2 - \sinh(3t)$

Solution:

(a) 0

(b) $\frac{e^{-3s}}{s}$

(c) $\frac{\Gamma(4/3)}{s^{4/3}}$

(d) $\frac{1}{s+2}$

(e) $\frac{2}{s} - \frac{3}{s^2-9}$

2. (5 points) Write down $\mathcal{L}^{-1}(F(s))$ for the following functions $F(s)$ (no work needs to be shown for this problem and no partial credit will be given).

   (a) $F(s) = \frac{1}{s}$

   (b) $F(s) = \frac{4s-2}{s^2+25}$

   (c) $F(s) = \frac{6s}{s^2-16}$

   (d) $F(s) = \frac{1}{s^4}$

   (e) $F(s) = \frac{10}{s^2} - \frac{3}{s-1}$

Solution:

(a) 1

(b) $4\cos(5t) - \frac{2}{5}\sin(5t)$

(c) $\frac{6}{4}\cosh(4t)$

(d) $\frac{t^3}{6}$

(e) $10t - 3e^t$

3. (16 points) Calculate

$$\mathcal{L}^{-1}\left(\frac{-2}{s(s-4)}\right).$$

Solution:

We know that

$$\mathcal{L}^{-1}\left(\frac{-2}{(s-4)}\right) = -2e^{4t}.$$

Thus,

$$\mathcal{L}^{-1}\left(\frac{-2}{s(s-4)}\right) = \int_0^t -2e^{4\theta}d\theta = -2\left[e^{4\theta}/4\right]_{\theta=0}^{t} = -e^{4t}/2 + 1/2.$$

This problem can also be solved using partial fractions or using convolution. I think this method is the easiest one however.

4. (16 points) Consider the system of differential equations

$$x''(t) + 3x(t) + y'(t) = e^t,$$
$$x''(t) + y''(t) + 2y(t) = 0.$$

Solution: Write down a fourth order differential equation that $x(t)$ satisfies. Your answer should be of the form

$$x^{(4)}(t) + P(t)x'''(t) + Q(t)x''(t) + Rx'(t) + S(t)x(t) = F(t).$$

We rewrite this system using the D-notation.

$$(D^2 + 3)x(t) + Dy(t) = e^t,$$
$$D^2x(t) + (D^2 + 2)y(t) = 0.$$

We need to eliminate the y-variable, so we multiply the first equation by $(D^2 + 2)$ and the second equation by D, and then subtract. We then get

$$[(D^2 + 3)(D^2 + 2) - D^3]x(t) + [D(D^2 + 2) - (D^2 + 2)D]y(t) = (D^2 + 2)e^t.$$

Since $(D^2 + 3)(D^2 + 2) - D^3 = D^4 - D^3 + 5D^2 + 6$, and $D^2e^t = e^t$, translating back to the $'$ notation, we get

$$x^{(4)} - x'''(t) + 5x''(t) + 6x(t) = 3e^t$$

5. (16 points) Solve the initial value problem

$$x''(t) + 16x(t) = 0; x(0) = 3, x'(0) = 0.$$

using Laplace transforms. **You will get no credit for solving this problem using any other method.**

Solution: Let us define $F(s) = \mathcal{L}(x(t))$. We have $\mathcal{L}(x''(t)) = s^2F(s) - 3s$, so taking Laplace transforms of the entire differential equation we get

$$s^2F(s) - 3s + 16F(s) = 0,$$

and so

$$F(s) = \frac{3s}{s^2 + 16.}$$

We then have

$$x(t) = \mathcal{L}^{-1}(F(s)) = 3\cos(4t).$$

6. (16 points) Calculate the Laplace transform

$$\mathcal{L}\left(\frac{1-e^{2t}}{t}\right).$$

You should first verify that the function makes sense at $t = 0$.

Solution: Using L'Hopital's rule, we have

$$\lim_{t\to 0}\frac{1-e^{2t}}{t} = \lim_{t\to 0}\frac{-2e^{2t}}{1} = -2.$$

We know that

$$\mathcal{L}(1-e^{2t}) = \frac{1}{s} - \frac{1}{s-2}.$$

We then have

$$\begin{aligned}
\mathcal{L}(1-e^{2t}) &= \int_s^\infty \frac{1}{\sigma} - \frac{1}{\sigma-2} \\
&= [\ln(\sigma) - \ln(\sigma-2)]_{\sigma=s}^\infty \\
&= \left[\ln\left(\frac{\sigma}{\sigma-2}\right)\right]_{\sigma=s}^\infty \\
&= \lim_{\sigma\to\infty}\ln\left(\frac{\sigma}{\sigma-2}\right) - \ln\left(\frac{s}{s-2}\right) \\
&= \ln\left(\lim_{\sigma\to\infty}\frac{\sigma}{\sigma-2}\right) - \ln\left(\frac{s}{s-2}\right) \\
&= \ln(1) - \ln\left(\frac{s}{s-2}\right) \\
&= -\ln\left(\frac{s}{s-2}\right).
\end{aligned}$$

7. (16 points)

(a) Calculate the inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}\left(\frac{e^{-2s}}{(s+1)^2}\right).$$

(b) What is the value of $f(1)$? Your answer should be a number.

Solution: We know that

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t.$$

Using the s-translation formula, we then have that

$$\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) = te^{-t}.$$

Finally, using the t-translation formula, we get that

$$f(t) = \mathcal{L}^{-1}\left(\frac{e^{-2s}}{(s+1)^2}\right) = u(t-2)[(t-2)e^{-(t-2)}].$$

Since $u(-1) = 0$, we must have that $f(1) = 0$.

8. (10 points) Is it true that if $\mathcal{L}(f(t)) = F(s)$, $\mathcal{L}(g(t)) = G(s)$

$$\mathcal{L}^{-1}\left(F(s)G(s)\right) = f(t)g(t)?$$

If it is true, prove it. If it is false, provide a counterexample.

Solution: Take $f(t) = g(t) = 1$. Then $F(s) = G(s) = 1/s$. However

$$\mathcal{L}^{-1}\left(\frac{1}{s}\frac{1}{s}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t \neq 1.$$