MATH 3113-009 Test III

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Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. No calculators allowed.

Name: ____

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	

Formula Sheet

$f(t)$	$\mathcal{L}(f(t))$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^a (a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$

The above Laplace transforms are valid for $s > 0$, with the exception of the Laplace transforms of $\sinh(kt)$ and $\cosh(kt)$ which are valid for $s > |k|$ and the Laplace transform of e^{at} which is valid for $s > a$.

x	$\Gamma(x)$
$n + 1$ (for integer $n \geq 0$)	$n!$
$1/2$	$\sqrt{\pi}$
$3/2$	$\frac{1}{2}\sqrt{\pi}$
$5/2$	$\frac{3}{4}\sqrt{\pi}$
$7/2$	$\frac{15}{8}\sqrt{\pi}$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt$$

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$$

$$\mathcal{L}(f^{(n)}(t)) = s^n \mathcal{L}(f(t)) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$$

$$\mathcal{L}\left(\int_0^t f(\theta) d\theta\right) = \frac{\mathcal{L}(f(t))}{s}$$

$$F(s) = \mathcal{L}(f(t)) \implies F(s-a) = \mathcal{L}(e^{at} f(t))$$

$$f(t) * g(t) = \int_0^t f(\theta) g(t-\theta) d\theta$$

$$\mathcal{L}(-t f(t)) = \frac{dF(s)}{ds}$$

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\sigma) d\sigma$$

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1. (5 points) Write down $\mathcal{L}(f(t))$ for the following functions $f(t)$ (no work needs to be shown for this problem).

   (a) $f(t) = 0$

   (b) $f(t) = \sin(t)$

   (c) $f(t) = \sqrt{t}$

   (d) $f(t) = e^{-t}$

   (e) $f(t) = 15 - 2\cosh(\pi t)$

Solution:

(a) 0

(b) $1/(s^2 + 1)$

(c) $\Gamma(3/2)/\sqrt{s^3}$

(d) $1/(s + 1)$

(e) $15/s - 2s/(s^2 - \pi^2)$

2. (5 points) Write down $\mathcal{L}^{-1}(F(s))$ for the following functions $F(s)$ (no work needs to be shown for this problem).

   (a) $F(s) = 0$

   (b) $F(s) = \frac{2s+3}{s^2+4}$

   (c) $F(s) = \frac{6}{s^2-4}$

   (d) $F(s) = \frac{1}{s^6}$

   (e) $F(s) = \frac{17}{s} - \frac{1}{s-17}$

Solution:

(a) 0

(b) $\cos(2t) + \sin(2t)$

(c) $3\sinh(2t)$

(d) $t^5/120$

(e) $17 - e^{17t}$

3. (20 points) According to the formula sheet, the Laplace transform of $f(t) = 1$ is $1/s$ for $s > 0$. Show that this is true using the definition of the Laplace transform.

Solution: Since $f(t) = 1$, by the definition of the Laplace transform,

$$\begin{aligned}
\mathcal{L}(1) &= \int_0^\infty 1e^{-st}dt \\
&= \left[\frac{e^{-st}}{-s}\right]_{t=0}^{\infty} \\
&= \left(\lim_{t\to\infty} \frac{e^{-st}}{-s}\right) - \frac{1}{-s} \\
&= \frac{1}{s}.
\end{aligned}$$

The limit $\lim_{t\to\infty} e^{-st} = 0$ since we assumed that $s > 0$.

4. (20 points) Consider the system of differential equations

$$x''(t) + x(t) + y''(t) = 2e^{-t},$$
$$x''(t) - x(t) + y''(t) = 0.$$

Using the operational determinant method, we can get the following candidate for the general solution: (you don't have perform this calculation)

$$x(t) = C_1 + C_2 t + e^{-t},$$
$$y(t) = D_1 + D_2 t,$$

for arbitrary constants C_1, C_2, D_1, D_2.

(a) Explain why the given candidate solution cannot be the general solution of the system.

(b) What is the correct general solution?

Solution: We rewrite this system as

$$(D^2 + 1)x(t) + D^2 y(t) = 2e^{-t},$$
$$(D^2 - 1)x(t) + D^2 y(t) = 0.$$

The operational matrix of this system is then

$$\begin{pmatrix} D^2 + 1 & D^2 \\ D^2 - 1 & D^2 \end{pmatrix},$$

and the operational determinant must be $D^2(D^2 + 1) - D^2(D^2 - 1) = 2D^2$. Since the determinant is a polynomial of degree 2, the general solution has 2 constants. So the candidate solution is has too many arbitrary constants, and we have to reduce them.

Plugging in the candidate solution to the first equation, we have $x''(t) = -e^{-t}$ and $y''(t) = 0$, and so

$$2e^{-t} + C_1 + C_2 t = 2e^{-t},$$

which implies

$$C_1 + C_2 t = 0.$$

This is only possible if $C_1 = C_2 = 0$. Thus we have reduced our solution to two constants,

$$x(t) = e^{-t},$$
$$y(t) = D_1 + D_2 t.$$

To see the full problem worked out, please visit
https://www.youtube.com/watch?v=WzJ5KwbnS8o

5. (20 points) Solve the initial value problem

$$x''(t) + 4x(t) = 2; x'(0) = x(0) = 0.$$

using Laplace transforms. **You will get no credit for solving this problem using any other method.**

Solution: Let $\mathcal{L}(x(t)) = X(s)$. We know that

$$\mathcal{L}(x''(t)) = s^2X(s) - sx(0) - x'(0) = s^2X(s).$$

Thus taking the Laplace transform of the differential equation we obtain

$$\begin{aligned} \mathcal{L}(x''(t)) + 4\mathcal{L}(x(t)) &= \mathcal{L}(2) \\ s^2X(s) + 4X(s) &= \frac{2}{s} \\ X(s) &= \frac{2}{s(s^2+4)}. \end{aligned}$$

Notice that $\mathcal{L}^{-1}(2/(s^2+4)) = \sin(2t)$. Hence, by the integration formula,

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{1}{s}\frac{2}{s^2+4}\right) &= \int_0^t \sin(2\theta)d\theta \\ &= \left[-\frac{1}{2}\cos(2\theta)\right]_{\theta=0}^{t} \\ &= -\frac{\cos(2t)}{2} + \frac{1}{2}. \end{aligned}$$

6. (20 points) Suppose that $X(s)$ satisfies the following equation:

$$(s^2+s)X'(s)+2sX(s)=0.$$

Find all possible $x(t)$ for which

$$\mathcal{L}^{-1}(X(s))=x(t).$$

Hint: what is the derivative of $\ln(X(s))$*?*

Solution: We can write

$$\frac{X'(s)}{X(s)}=\frac{-2s}{s^2+s}=\frac{-2}{s+1}$$

Since $\frac{d}{ds}\ln(X(s))=X'(s)/X(s)$, we must have

$$\begin{aligned}
\frac{d}{ds}\ln(X(s)) &= \frac{-2}{s+1}\\
\ln(X(s)) &= \int\frac{-2}{s+1}ds\\
&= -2\ln(s+1)+C\\
X(s) &= e^{-2\ln(s+1)+C}\\
&= e^{-2\ln(s+1)}e^{C}\\
&= (e^{\ln(s+1)})^{-2}e^{C}\\
&= (s+1)^{-2}e^{C}\\
&= \frac{e^{C}}{(s+1)^2}
\end{aligned}$$

Note that since $\mathcal{L}^{-1}(1/s^2)=t$, by the translation formula

$$\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right)=te^{-t}.$$

And so finally,

$$x(t)=\mathcal{L}^{-1}(X(s))=e^{C}\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right)=e^{C}te^{-t}.$$

7. (10 points) Let $f(t)$ be a function for which a Laplace transform $\mathcal{L}(f(t))$ exists for $s > c$, for some constant c. Is it true that

$$\mathcal{L}\left(f(t)^2\right) = \mathcal{L}(f(t))^2?$$

If it is true, derive this formula using the definition of the Laplace transform. If it is not true, use a counterexample to show that the formula is false.

Solution: The statement is false. As a counterexample, take $f(t) = 1$. The LHS becomes $1/s$, and the RHS becomes $1/s^2$. There are also several other possible counterexamples.