MATH 3113-008 Test III

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> Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. No calculators allowed.

Name: ____

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	

Formula Sheet

$f(t)$	$\mathcal{L}(f(t))$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^a(a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$

The above Laplace transforms are valid for $s > 0$, with the exception of the Laplace transforms of $\sinh(kt)$ and $\cosh(kt)$ which are valid for $s > |k|$ and the Laplace transform of e^{at} which is valid for $s > a$.

x	$\Gamma(x)$
$n+1$ (for integer $n \geq 0$)	$n!$
$1/2$	$\sqrt{\pi}$
$3/2$	$\frac{1}{2}\sqrt{\pi}$
$5/2$	$\frac{3}{4}\sqrt{\pi}$
$7/2$	$\frac{15}{8}\sqrt{\pi}$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt$$

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$$

$$\mathcal{L}(f^{(n)}(t)) = s^n \mathcal{L}(f(t)) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$$

$$\mathcal{L}\left(\int_0^t f(\theta) d\theta\right) = \frac{\mathcal{L}(f(t))}{s}$$

$$F(s) = \mathcal{L}(f(t)) \implies F(s-a) = \mathcal{L}(e^{at} f(t))$$

$$f(t) * g(t) = \int_0^t f(\theta) g(t-\theta) d\theta$$

$$\mathcal{L}(-t f(t)) = \frac{dF(s)}{ds}$$

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\sigma) d\sigma$$

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1. (5 points) Write down $\mathcal{L}(f(t))$ for the following functions $f(t)$ (no work needs to be shown for this problem).

   (a) $f(t) = 0$

   (b) $f(t) = \sin(t)$

   (c) $f(t) = \sqrt{t}$

   (d) $f(t) = e^{-t}$

   (e) $f(t) = 15 - 2\cosh(\pi t)$

Solution:

(a) 0

(b) $1/(s^2 + 1)$

(c) $\Gamma(3/2)/\sqrt{s^3}$

(d) $1/(s + 1)$

(e) $15/s - 2s/(s^2 - \pi^2)$

2. (20 points) Calculate

$$\mathcal{L}^{-1}\left(\frac{2}{(s+3)(s-1)}\right).$$

Solution: We use partial fractions to determine that

$$\begin{aligned} frac2(s+3)(s-1) =& \frac{A}{s+3} + \frac{B}{s-1}, \\ 2 =& A(s-1) + B(s+3), \end{aligned}$$

Plugging in $s = 1$ we obtain $3B = 2$, and so $B = 1/2$. Plugging in $s = -3$ we obtain $-4A = 2$, and so $A = -1/2$. We conclude that

$$L^{-1}\left(\frac{2}{(s+3)(s-1)}\right) = \mathcal{L}\left(\frac{-1/2}{s+3}\right) + \mathcal{L}^{-1}\left(\frac{1/2}{s-1}\right) = -\frac{1}{2}e^{-3t} + \frac{1}{2}e^{t}.$$

3. (20 points) Consider the system of differential equations. You do not have to solve the system.

$$\begin{aligned} x''(t) + y''(t) + x(t) &= t^2, \\ x''(t) + y''(t) + y'(t) + y(t) &= 0. \end{aligned}$$

(a) What is the operational determinant of this system?

(b) How many independent arbitrary constants do you expect to find in its general solution?

Solution: We rewrite the system as

$$\begin{aligned} (D^2 + 1)x(t) + D^2 y(t) &= t^2, \\ D^2 x(t) + (D^2 + D + 1)y(t) &= 0. \end{aligned}$$

The operational determinant is thus

$$(D^2 + 1)(D^2 + D + 1) - D^2 D^2 = D^3 + 2D^2 + D + 1.$$

The degree of the operational determinant is 3, and so we expect there to be 3 independent constants in the general solution.

4. (20 points) Consider the following two initial value problems.

(a)

$$x''(t) + 6x'(t) + 19x = \cos(2t), x(0) = 1, x'(0) = -1.$$

Calculate $\mathcal{L}(x(t))$. You do not need to find the solution $x(t)$.

(b)

$$ty''(t) + (2t-3)y'(t) + (t-1)y(t) = 0, y(0) = 0.$$

Where $Y(s) = \mathcal{L}(y(t))$, calculate $\frac{d\ln(Y(s))}{ds}$. You do not need to find the solution $y(t)$.

Solution:

(a) We have

$$\begin{aligned}\mathcal{L}(x(t)) =& X(s),\\ \mathcal{L}(x'(t)) =& sX(s) - x(0) = sX(s) - 1\\ \mathcal{L}(x''(t)) =& s^2X(s) - sx(0) - x'(0) = s^2X(s) - s + 1.\end{aligned}$$

And so

$$\begin{aligned}\mathcal{L}(x''(t) + 6x'(t) + 19x) =& \mathcal{L}(\cos(2t))\\ s^2X(s) - s + 1 + 6(sX(s) - 1) + 19X(s) =& \frac{s}{s^2+4}.\end{aligned}$$

Solving for $X(s)$, we conclude that

$$X(s) = \frac{\frac{s}{s^2+4} + 5 + s}{s^2 + 6s + 19}.$$

(b) We have

$$\begin{aligned}\mathcal{L}(y(t)) =& Y(s),\\ \mathcal{L}(y'(t)) =& sY(s) - y(0) = sY(s)\\ \mathcal{L}(y''(t)) =& s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - y'(0).\\ \mathcal{L}(ty(t)) =& -\frac{d}{ds}Y(s),\\ \mathcal{L}(ty'(t)) =& -\frac{d}{ds}(sY(s) - y(0)) = -\frac{d}{ds}(sY(s)) = -Y(s) - sY'(s)\\ \mathcal{L}(ty''(t)) =& -\frac{d}{ds}(s^2Y(s) - sy(0) - y'(0)) = -\frac{d}{ds}(s^2Y(s) - y'(0)) = -2sY(s) - s^2Y'(s).\end{aligned}$$

Taking the Laplace transform of the differential equation, we have

$$\begin{aligned}
\mathcal{L}(ty''(t)) + 2\mathcal{L}(ty'(t)) - 3\mathcal{L}(y'(t)) + \mathcal{L}(ty(t)) - \mathcal{L}(y(t)) =& 0. \\
(-2sY(s) - s^2Y'(s)) + 2(-Y(s) - sY'(s)) - 3sY(s) - Y'(s) - Y(s) =& 0 \\
Y'(s)(-s^2 - 2s - 1) + Y(s)(-2s - 2 - 3s - 1) =& 0 \\
\frac{d}{ds}\ln(Y(s)) = \frac{Y'(s)}{Y(s)} =& -\frac{5s+3}{s^2+2s+1}
\end{aligned}$$

5. (20 points) Calculate

$$\mathcal{L}\left(\frac{e^t - e^{-t}}{t}\right).$$

Solution: Since

$$\mathcal{L}(e^t - e^{-t}) = \frac{1}{s-1} - \frac{1}{s+1},$$

We have

$$\begin{aligned}
\mathcal{L}\left(\frac{e^t - e^{-t}}{t}\right) &= \int_s^\infty \frac{1}{\sigma-1} - \frac{1}{\sigma+1} d\sigma \\
&= \ln(\sigma-1) - \ln(\sigma+1)|_{\sigma=s}^\infty \\
&= \ln\left(\frac{\sigma-1}{\sigma+1}\right)\Big|_{\sigma=s}^\infty \\
&= \lim_{\sigma\to\infty} \ln\left(\frac{\sigma-1}{\sigma+1}\right) - \ln\left(\frac{s-1}{s+1}\right) \\
&= -\ln\left(\frac{s-1}{s+1}\right).
\end{aligned}$$

The reason that

$$\lim_{\sigma\to\infty} \ln\left(\frac{\sigma-1}{\sigma+1}\right) = 0$$

is due to L'Hopital's rule, but you will not be penalized for not mentioning that.

6. (10 points) According to your formula sheet, the Laplace transform $\mathcal{L}(e^{at}) = \frac{1}{s-a}$ is valid for $s > a$. Explain why it isn't valid for $s < a$.

Solution: There are at least three possible solutions.

1. Using the definition of the Laplace transform,

$$\begin{aligned}\mathcal{L}(e^{at}) &= \int_0^\infty e^a t e^{-st} dt \\ &= \int_0^\infty e^{at-st} dt \\ &= \int_0^\infty e^{(a-s)t} dt \\ &= \frac{e^{(a-s)t}}{a-s}\bigg|_{t=0}^\infty \\ &= \lim_{t\to\infty} e^{(a-s)t} - \frac{1}{a-s}.\end{aligned}$$

   If $s < a$, the limit will go to infinity.

2. If $s < a$ and $\mathcal{L}(e^{at}) = 1/(s-a)$, it must be true that $L(e^{at})$ is negative. However,

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{at} e^{-st} dt$$

   Since both e^{at} and e^{-st} are positive, the integral must be positive too, which is a contradiction.

3. An answer using Theorem 2 of section 7.1 is also acceptable.