1. (10 points) Obtain the forward and backward difference approximations of $f'(x_0)$ from the Taylor series,

$$f(x_0 + \Delta x) \approx f(x_0) + \frac{\Delta x}{1!}\frac{d}{dx}f(x_0) + \frac{(\Delta x)^2}{2!}\frac{d^2}{dx^2}f(x_0) + \ldots + \frac{(\Delta x)^n}{n!}\frac{d^n}{dx^n}f(x_0).$$

You do not have to calculate the error.

$$f(x_0+\Delta x) \approx f(x_0) + \Delta x\, f'(x_0).$$

Solve for $f'(x_0)$:

$$f'(x_0) \approx \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$ Forward diff approximation

Replace ~~Set~~ Δx with $-\Delta x$

$$f'(x_0) \approx \frac{f(x_0-\Delta x) - f(x_0)}{-\Delta x}$$

$$= \frac{f(x_0) - f(x_0-\Delta x)}{\Delta x}$$ Backward diff approximation

2. (15 points) Consider the following eigenvalue problem.

$$F''(x) + (\lambda - x^2)F(x) = 0, F'(0) = F(1) = 0.$$

The Rayleigh quotient of this equation is

$$RQ[u] = \frac{\int_0^1 (F'(x))^2 + x^2 F(x)^2)dx}{\int_0^1 F(x)^2 dx}.$$

Use the minimization principle to obtain a reasonably accurate upper bound for the lowest eigenvalue of this eigenvalue problem. (The minimization principle states that the lowest eigenvalue λ_1 satisfies $\lambda_1 = \min RQ[u]$, where the minimum runs through all continuous functions $u(x)$ satisfying the boundary conditions)

Let $u(x)$ be defined as $u(x) = x^2 - 1$

It satisfies both boundary conditions.

$$\lambda_1 \leq \frac{\int_0^1 (u'(x))^2 + x^2 u(x)^2 \, dx}{\int_0^1 (u(x))^2 dx}$$

$$= \frac{\int_0^1 4x^2 + x^2(x^2-1)^2 dx}{\int_0^1 (x^2-1)^2 dx}.$$

Denominator $= \int_0^1 x^4 + 1 - 2x^2 \, dx = \frac{x^5}{5} + x - \frac{2}{3}x^3 \Big|_0^1 = \frac{1}{5} + 1 - \frac{2}{3} = \frac{8}{15}$

Numerator $= \int_0^1 4x^2 + x^2(x^4 - 2x^2 + 1) \, dx$

$= \int_0^1 x^6 - 2x^4 + 5x^2 \, dx$

$= \frac{x^7}{7} - \frac{2x^5}{5} + \frac{5x^3}{3} \Big|_0^1 = \frac{1}{7} - \frac{2}{5} + \frac{5}{3}$

$= \frac{15 - 42 + 175}{105} = \frac{148}{105} =$

$$\lambda_1 \leq \frac{\frac{148}{105}}{\frac{8}{15}} = \frac{37}{2} \cdot \frac{1}{7} = \frac{37}{14}$$

3. (15 points) Find all the eigenvalues for the following two-dimensional eigenvalue problem:

$$\frac{\partial^2}{\partial x^2}F(x,y) + \frac{\partial^2}{\partial y^2}F(x,y) = -\lambda F(x,y),$$

with boundary conditions

$$\frac{\partial}{\partial y}F(0,y) = \frac{\partial}{\partial y}F(L,y) = \frac{\partial}{\partial x}F(x,0) = \frac{\partial}{\partial x}F(x,H) = 0$$

Hint 1: You might want to use $F(x,y) = f(x)g(y)$ to reduce this problem into two one-dimensional eigenvalue problems.

Hint 2: The 1-D eigenvalue problem $f'(x) + \mu f(x) = 0$ with boundary conditions $f'(0) = f'(L) = 0$ has eigenvalues $0, (\pi/L)^2, \ldots, (\pi n/L)^2, \ldots$

$$f''(x)\cdot g(y) + f(x)g''(y) = -\lambda f(x)g(y)$$

$$\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} = -\lambda$$

$$\frac{f''(x)}{f(x)} = -\lambda - \frac{g''(y)}{g(y)} = -\mu \quad \text{a constant}$$

$f'(0) = f'(L) = 0$ $\quad f''(x) + \mu f(x) = 0$ has e-values $\mu_n = \left(\frac{\pi n}{L}\right)^2$, $n = 0, 1, \ldots$

Need to solve

$$-\lambda - \frac{g''(y)}{g(y)} = -\left(\frac{\pi n}{L}\right)^2.$$

$g'(0) = g'(L) = 0$

$$g''(y) + \underbrace{\left(\lambda - \left(\frac{\pi n}{L}\right)^2\right)}_{\text{treat as constant } k} g(y) = 0$$

$g''(y) + k\,g(y) = 0$

has e-values

$$k = \left(\frac{\pi m}{H}\right)^2, \quad m = 0, 1, 2, \ldots$$

$$\left(\lambda - \left(\frac{\pi n}{L}\right)^2\right) = \left(\frac{\pi m}{H}\right)^2$$

$$\lambda_{nm} = \left(\frac{\pi n}{L}\right)^2 + \left(\frac{\pi m}{H}\right)^2 \quad n = 0, 1, 2, \ldots \quad m = 0, 1, 2, \ldots$$

4. (15 points) Let $z \geq 0$. The Bessel equation

$$z^2 \frac{d^2}{dz^2} f(\sqrt{\lambda} z) + z \frac{d}{dz} f(\sqrt{\lambda} z) + (z^2 - m^2) f(\sqrt{\lambda} z) = 0$$

has two linearly independent solutions $J_m(\sqrt{\lambda} z)$ and $Y_m(\sqrt{\lambda} z)$. It is known that $J_m(z)$ is a continuous function for all z, whereas $Y_m(z)$ tends to $-\infty$ as z tends to 0. Let the positive roots of $J_m(z)$ be given as $\zeta(J_m, 1)$, $\zeta(J_m, 2)$, $\zeta(J_m, 3)$, ... and the positive roots of $Y_m(z)$ be given as $\zeta(Y_m, 1)$, $\zeta(Y_m, 2)$, $\zeta(Y_m, 3)$, ...

Find all the eigenvalues λ of the eigenvalue problem given by the Bessel equation, together with the boundary value $f(\sqrt{\lambda} L) = 0$ and the condition $|f(0)| < \infty$.

General solution is

$$f(z) = c_1 J_m(\sqrt{\lambda} z) + c_2 Y_m(\sqrt{\lambda} z)$$

plug $z=0$ $\quad f(0) = c_1 J_m(0) + c_2 Y_m(0)$. $\quad |f(0)| < \infty$

J_m is continuous, therefore $J_m(0) < \infty$

~~$J_m(0)$ continuous, +~~ $\quad$ & $Y_m(z) \to -\infty$ as $z \to 0$

only possible if $c_2 = 0$

$$f(z) = c_1 J_m(z).$$

solution is nontrivial only if $c_1 \neq 0$.

plug in boundary condition

$\cancel{0 = f(L) = c_1 J_m(L)}$. $\quad 0 = f(\sqrt{\lambda} L) = c_1 J_m(\sqrt{\lambda} L) = 0$

Need $J_m(L) = 0$. so $L =$ $\quad$ $c_1 \neq 0$ only if $J_m(\sqrt{\lambda} L) = 0$

$J_m(\sqrt{\lambda} L) = 0$ only if

$\sqrt{\lambda} L = \zeta(J_m, n)$ for some n

$$\boxed{\text{so } \lambda_n = \left(\frac{\zeta(J_m, n)}{L}\right)^2}$$

5. (15 points) We say that an equation is homogeneous if it can be expressed in the form $\mathcal{L}(u) = 0$, where $\mathcal{L}$ is an operator that behaves "nicely" with respect to addition and constant multiplication.

Determine whether or not the following two equations are homogeneous, by first identifying the operator $\mathcal{L}$ and checking if $\mathcal{L}$ behaves nicely with respect to addition and constant multiplication.

(a) (Vibrating string equation with air resistance)

$$\frac{\partial^2}{\partial t^2}u(x,t) = c\frac{\partial^2}{\partial x^2}u(x,t) - r\frac{\partial}{\partial t}u(x,t),$$

with c, r both constants.

~~(b) (1-D heat equation~~ with constant heat generation)

$$\cancel{\frac{\partial}{\partial t}u(x,t) = k\frac{\partial^2}{\partial x^2}u(x,t) + Q,}$$

where ~~Q is a constant.~~

(a) $\mathcal{L} = \frac{\partial^2}{\partial t^2} - c\frac{\partial^2}{\partial x^2} + r\frac{\partial}{\partial t}$:

$$\begin{aligned}\mathcal{L}(\alpha u(x,t) + \beta v(x,t)) &= \frac{\partial^2}{\partial t^2}(\alpha u(x,t) + \beta v(x,t)) - c\frac{\partial^2}{\partial x^2}(\alpha u(x,t) + \beta v(x,t)) \\ &\quad + r\frac{\partial}{\partial t}(\alpha u(x,t) + \beta v(x,t)) \\ &= \alpha\left(\frac{\partial^2}{\partial t^2}u(x,t) - c\frac{\partial^2}{\partial x^2}u(x,t) + r\frac{\partial}{\partial t}u(x,t)\right) \\ &\quad + \beta\left(\frac{\partial^2}{\partial t^2}v(x,t) - c\frac{\partial^2}{\partial x^2}v(x,t) + r\frac{\partial}{\partial t}v(x,t)\right) \\ &= \alpha\mathcal{L}(u) + \beta\mathcal{L}(v)\end{aligned}$$

6. (15 points) Consider the following nonhomogeneous partial differential equation with nonhomogeneous boundary conditions:

$u(0,t) = t, \quad u(5,t) = t^2$

$$\frac{\partial^2}{\partial t^2}u(x,t) = c\frac{\partial^2}{\partial x^2}u(x,t) - \sin(x+t), \frac{\partial}{\partial t}\cancel{u(0,t) - u(5,}t) = 0$$

where c is a constant.

Use a change of variable to convert it to a nonhomogeneous partial differential equation with homogeneous boundary conditions. Hint: try to find a function $r(x,t)$ that satisfies the boundary conditions.

$r(0,t) = t, \quad r(5,t) = t^2$

$r(x,t) = t + \frac{x(t^2 - t)}{5}$

change of variable $v(x,t) = u(x,t) - r(x,t)$, $u(x,t) = v(x,t) + t + \frac{x(t^2-t)}{5}$

$v(0,t) = 0, \quad v(5,t) = 0$

$\frac{\partial}{\partial t} v(x,t) + 2x = c\frac{\partial^2}{\partial x^2} v(x,t) - \sin(x+t).$

7. (15 points) Consider the partial difference equation given by

$$u_j^{(m+1)} = u_j^{(m)} + s(u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}),$$

with boundary conditions

$$u_0^{(m)} = u_6^{m} = 0,$$

and intial condition

$$u_j^{(0)} = \begin{cases} 0, j \text{ even}, \\ 1, j \text{ odd}. \end{cases}$$

Set $s = \frac{1}{4}$, and find $u_2^{(2)}$.

$$u_2^{(2)} = u_2^{(1)} + \frac{1}{4}\left(u_3^{(1)} - 2u_2^{(1)} + u_1^{(1)}\right)$$

$$u_3^{(1)} = u_3^{(0)} + \frac{1}{4}\left(u_4^{(0)} - 2u_3^{(1)} + u_1^{(0)}\right)$$

$$= 1 + \frac{1}{4}(0 - 2 + 0)$$

$$= 0.5$$

$$u_2^{(1)} = u_2^{(0)} + \frac{1}{4}\left(u_3^{(0)} - 2u_2^{(0)} + u_1^{(0)}\right)$$

$$= 0 + \frac{1}{4}(1 - 0 + 1)$$

$$= 0.5$$

$$u_1^{(1)} = u_1^{(0)} + \frac{1}{4}\left(u_2^{(0)} - 2u_1^{(0)} + u_0^{(0)}\right)$$

$$= 1 + \frac{1}{4}(0 - 2 + 0)$$

$$= 0.5$$

$$u_2^{(2)} = 0.5 + \frac{1}{4}(0.5 - 2(0.5) + 0.5)$$

$$= 0.5$$