Formula Sheet

$f(t)$	$\mathcal{L}(f(t))$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^a (a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$

The above Laplace transforms are valid for $s > 0$, with the exception of the Laplace transforms of $\sinh(kt)$ and $\cosh(kt)$ which are valid for $s > |k|$ and the Laplace transform of e^{at} which is valid for $s > a$.

x	$\Gamma(x)$
$n + 1$ (for integer $n \geq 0$)	$n!$
$1/2$	$\sqrt{\pi}$
$3/2$	$\frac{1}{2}\sqrt{\pi}$
$5/2$	$\frac{3}{4}\sqrt{\pi}$
$7/2$	$\frac{15}{8}\sqrt{\pi}$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt$$

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$$

$$\mathcal{L}(f^{(n)}(t)) = s^n \mathcal{L}(f(t)) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$$

$$\mathcal{L}\left(\int_0^t f(\theta)d\theta\right) = \frac{\mathcal{L}(f(t))}{s}$$

$$F(s) = \mathcal{L}(f(t)) \implies F(s-a) = \mathcal{L}(e^{at} f(t))$$

$$f(t) * g(t) = \int_0^t f(\theta)g(t-\theta)d\theta$$

$$\mathcal{L}(-tf(t)) = \frac{dF(s)}{ds}$$

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\sigma)d\sigma$$

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MATH 3113-002 Test III

Dr. Darren Ong

December 4, 2015 11:30am-12:20am

Answer the questions in the spaces provided on the question sheet. No calculators allowed.

Name: ______________________________

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

1. (10 points) Solve the following Laplace transforms or inverse Laplace transforms. Please show your work, as there is the possibility of partial credit.

   (a) $\mathcal{L}(e^{-3t/2})$

   (b) $\mathcal{L}(2)$

   (c) $\mathcal{L}(2\sin(t) - \pi\sqrt{t})$

   (d) $\mathcal{L}^{-1}\left(\frac{1-s}{s^2+36}\right)$

   (e) $\mathcal{L}^{-1}\left(\frac{1}{\sqrt{s^5}}\right)$

Solution:

(a) 1/(s+3/2)

(b) 2/s

(c) $\frac{2}{s^2+1} - \pi\frac{\Gamma(3/2)}{s^{3/2}}$

(d) $\frac{1}{6}\sin(6t) - \cos(6t)$

(e) $\frac{t^{3/2}}{\Gamma(3/2)}$

2. (20 points) Calculate the Laplace transform

$$\mathcal{L}(t\sin(t)).$$

Solution:

We have

$$\begin{aligned}\mathcal{L}(\sin(t)) &= \frac{1}{s^2+1}\\ \mathcal{L}(t\sin(t)) &= -\frac{d}{ds}\frac{1}{s^2+1}\\ &= \frac{2s}{(s^2+1)^2}\end{aligned}$$

3. (20 points) Calculate the inverse Laplace transform

$$\mathcal{L}^{-1}\left(\frac{2s-1}{s^2+6s+13}\right).$$

Hint: you might want to complete the square in the denominator.

Solution: Completing the square in the denominator, we rewrite the problem as

$$\mathcal{L}^{-1}\left(\frac{2s-1}{(s+3)^2+4}\right).$$

We can rewrite this as

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{2(s+3)-1-6}{(s+3)^2+4}\right) &= \mathcal{L}^{-1}\left(\frac{2(s+3)}{(s+3)^2+4}\right) - \mathcal{L}^{-1}\left(\frac{7}{(s+3)^2+4}\right) \\ &= 2e^{-3t}\cos(2t) - \frac{7}{2}e^{-3t}\sin(2t).\end{aligned}$$

4. (20 points) Solve the initial value problem

$$x''(t) - 9x(t) = 0; x(0) = 0, x'(0) = 2.$$

using Laplace transforms. **You will get no credit for solving this problem using any other method.**

Solution: We have

$$\begin{aligned}\mathcal{L}(x(t)) =& X(s),\\ \mathcal{L}(x'(t)) =& sX(s) - x(0) = sX(s)\\ \mathcal{L}(x''(t)) =& s^2X(s) - sx(0) - x'(0) = s^2X(s) - 2.\end{aligned}$$

So taking the Laplace transform of the equation gets us

$$s^2X(s) - 2 - 9X(s) = 0,$$

which is equivalent to

$$X(s) = \frac{2}{s^2 - 9}.$$

Thus

$$x(t) = \mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1}\left(\frac{2}{s^2 - 9}\right) = \frac{2}{3}\mathcal{L}^{-1}\left(\frac{3}{s^2 - 9}\right) = \frac{2}{3}\sinh(3t).$$

5. (20 points) Calculate the inverse Laplace transform

$$\mathcal{L}^{-1}\left(\ln\left(\frac{s}{s+3}\right)\right).$$

Hint: $\ln(A/B) = \ln(A) - \ln(B)$, and $\dfrac{d}{dx}\ln F(x) = F'(x)/F(x)$.

Solution:

We first find

$$\begin{aligned}
&\mathcal{L}^{-1}\left(\frac{d}{ds}\ln\left(\frac{s}{s+3}\right)\right)\\
=&\mathcal{L}^{-1}\left(\frac{d}{ds}\ln(s)\right) - \mathcal{L}^{-1}\left(\frac{d}{ds}\ln(s+3)\right)\\
=&\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{1}{s-3}\right)\\
=&1 - e^{3t}
\end{aligned}$$

We know that

$$\begin{aligned}
\mathcal{L}\left(\frac{1-e^{3t}}{t}\right) &= \int_s^\infty \frac{d}{d\sigma}\ln\left(\frac{\sigma}{\sigma+3}\right) d\sigma\\
&= \left[\ln\left(\frac{\sigma}{\sigma+3}\right)\right]_{\sigma=s}^{\infty}\\
&= \lim_{\sigma\to\infty}\ln\left(\frac{\sigma}{\sigma+3}\right) - \ln\left(\frac{s}{s+3}\right)\\
&= \ln(1) - \ln\left(\frac{s}{s+3}\right)\\
&= -\ln\left(\frac{s}{s+3}\right).
\end{aligned}$$

And so

$$\mathcal{L}^{-1}\left(\ln\left(\frac{s}{s+3}\right)\right) = -\frac{1-e^{3t}}{t}.$$

6. (10 points) Using the Laplace transform table, prove that

$$\sinh(2t) = \frac{e^{2t} - e^{-2t}}{2}.$$

Solution: We note that

$$\mathcal{L}(\sinh(2t)) = \frac{2}{s^2 - 4} = \frac{1/2}{s-2} - \frac{1/2}{s+2} = \mathcal{L}\left(\frac{e^{2t} - e^{-2t}}{2}\right).$$

We can now just take inverse Laplace transforms.