Formula Sheet

$f(t)$	$\mathcal{L}(f(t))$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^a (a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$

The above Laplace transforms are valid for $s > 0$, with the exception of the Laplace transforms of $\sinh(kt)$ and $\cosh(kt)$ which are valid for $s > |k|$ and the Laplace transform of e^{at} which is valid for $s > a$.

x	$\Gamma(x)$
$n+1$ (for integer $n \geq 0$)	$n!$
$1/2$	$\sqrt{\pi}$
$3/2$	$\frac{1}{2}\sqrt{\pi}$
$5/2$	$\frac{3}{4}\sqrt{\pi}$
$7/2$	$\frac{15}{8}\sqrt{\pi}$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt$$

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$$

$$\mathcal{L}(f^{(n)}(t)) = s^n \mathcal{L}(f(t)) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$$

$$\mathcal{L}\left(\int_0^t f(\theta) d\theta\right) = \frac{\mathcal{L}(f(t))}{s}$$

$$F(s) = \mathcal{L}(f(t)) \implies F(s-a) = \mathcal{L}(e^{at} f(t))$$

$$f(t) * g(t) = \int_0^t f(\theta) g(t-\theta) d\theta$$

$$\mathcal{L}(-t f(t)) = \frac{dF(s)}{ds}$$

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\sigma) d\sigma$$

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MATH 3113-001 Test III

Dr. Darren Ong

December 4, 2015 1:30pm-2:20pm

Answer the questions in the spaces provided on the question sheet. No calculators allowed.

Name: ____

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

1. (10 points) Calculate the following Laplace transforms or inverse Laplace transforms. Please show your work, as there is the possibility of partial credit.

   (a) $\mathcal{L}(e^{-3t/2})$
   (b) $\mathcal{L}(2)$
   (c) $\mathcal{L}(2\sin(t) - \pi\sqrt{t})$
   (d) $\mathcal{L}^{-1}\left(\frac{1-s}{s^2+36}\right)$
   (e) $\mathcal{L}(\sin(2t) * \cos(3t))$ (Hint: DON'T calculate the convolution product)

Solution:

(a) 1/(s+3/2)

(b) 2/s

(c) $\frac{2}{s^2+1} - \pi\frac{\Gamma(3/2)}{s^{3/2}}$

(d) $\frac{1}{6}\sin(6t) - \cos(6t)$

(e) $\frac{1}{s^2+4}\frac{s}{s^2+9}$

2. (20 points) Calculate the inverse Laplace transform

$$\mathcal{L}^{-1}\left(\frac{s+1}{s^2+5s+6}\right)$$

Solution: We factor the denominator and then we have the partial fraction decomposition

$$\frac{s+1}{(s+2)(s+3)} = \frac{2}{s+3} - \frac{1}{s+2}.$$

Thus

$$\mathcal{L}^{-1}\left(\frac{s+1}{s^2+5s+6}\right) = 2\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = 2e^{-3t} - e^{-2t}.$$

3. (20 points) Calculate the inverse Laplace transform

$$\mathcal{L}^{-1}\left(\frac{1}{(s-2)^{3/2}}\right)$$

Solution: We have, from the formula sheet

$$\mathcal{L}^{-1}\left(\frac{\Gamma(3/2)}{s^{3/2}}\right) = t^{1/2},$$

and so

$$\mathcal{L}^{-1}\left(\frac{1}{s^{3/2}}\right) = \frac{t^{1/2}}{\Gamma(3/2)},$$

and finally

$$\mathcal{L}^{-1}\left(\frac{1}{(s-2)^{3/2}}\right) = \frac{t^{1/2}e^{2t}}{\Gamma(3/2)}.$$

4. (20 points) Where x and y are functions of t, consider the system of equations

$$\begin{aligned} 2x''(t) + 3y'(t) + x(t) &= e^t \\ y''(t) + 2x'(t) - 3y(t) &= 0 \end{aligned}$$

It is known that $x(t)$ satisfies an equation of the form
$Ax^{(4)}(t) + Bx^{(3)}(t) + Cx''(t) + Dx'(t) + Ex(t) = Fe^t$,
where A, B, C, D, E, F are constants that are not all zero. Determine what A, B, C, D, E, F should be.

Solution: We rewrite the system as

$$\begin{aligned} (2D^2 + 1)x(t) + 3Dy(t) &= e^t \\ 2Dx(t) + (D^2 - 3)y(t) &= 0 \end{aligned}$$

Let (I) be the first equation and (II) the second equation. Then $(D^2 - 3)$(I)-$3D$(II) gives us

$$[(D^2 - 3)(2D^2 + 1) - 3D(2D)]x(t) = (D^2 - 3)e^t.$$

This is equivalent to

$$[2D^4 - 5D^2 - 3]x(t) = D^2e^t - 3e^t,$$

and so

$$2x^{(4)}(t) - 5x''(t) - 3x(t) = -2e^t.$$

So $A = -1, C = -5, E = -3, F = -2$, and $B = D = 0$.

5. (20 points) For the initial value problem

$$tx''(t) + tx'(t) + 2x'(t) + 2x(t) = 0, x(0) = 0.$$

Calculate $\frac{d}{ds}\ln(X(s))$, where $X(s) = \mathcal{L}(x(t))$.

Solution: We have

$$\begin{aligned}
\mathcal{L}(x(t)) =& X(s),\\
\mathcal{L}(x'(t)) =& sX(s) - x(0) = sX(s)\\
\mathcal{L}(x''(t)) =& s^2X(s) - sx(0) - x'(0) = s^2X(s) - x'(0).\\
\mathcal{L}(tx(t)) =& -\frac{d}{ds}X(s),\\
\mathcal{L}(tx'(t)) =& -\frac{d}{ds}(sX(s) - x(0)) = -\frac{d}{ds}(sX(s)) = -X(s) - sX'(s)\\
\mathcal{L}(tx''(t)) =& -\frac{d}{ds}(s^2X(s) - sx(0) - x'(0)) = -\frac{d}{ds}(s^2X(s) - x'(0)) = -2sX(s) - s^2X'(s).
\end{aligned}$$

So taking the Laplace transform of the entire differential equation we get

$$-2sX(s) - s^2X'(s) - X(s) - sX'(s) + 2sX(s) + 2X(s) = 0.$$

We can rewrite this as

$$X'(s)/X(s) = \frac{-1}{-s^2 - s}.$$

But since $\frac{d}{ds}\ln(X(s)) = X'(s)/X(s)$, we have

$$\frac{d}{ds}\ln(X(s)) = \frac{-1}{-s^2 - s}.$$

6. (10 points)

(a) Let k be any constant. Show that $e^{t^2} > e^{kt}$ for any t that is sufficiently large.

(b) By (a), we can say that $f(t) = e^{t^2}$ is not of exponential order. According to what we have learned in this class, this implies that $\mathcal{L}(f(t))$ does not exist. What goes wrong when you try to calculate the Laplace transform of $\mathcal{L}(f(t))$, using the definition of Laplace transform? (Hint: use the fact that $\frac{f(t)}{e^{st}} > 1$ for sufficiently large t.)

Solution: For part (a), we take the ln of both sides, so the inequality changes to $t^2 > kt$, which is obviously true for large enough t. This works because ln is an increasing function.

For part (b), we consider the integral

$$\mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-st}dt.$$

For large t, $f(t)e^{-st} = \frac{f(t)}{e^{st}} > 1$. Thus the area under the curve $f(t)e^{-st}$ contains a rectangle with height 1 and infinite length. The integral is thus infinity.