MATH 3113-170 Test II

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Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. No calculators allowed.

Name: ______________________________

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	

1. (15 points) A linear homogeneous third-order differential equation

$$y''' + 3y''(x) + 3y'(x) + y(x) = 0,$$

has solutions $y_1(x) = e^{-x}$, $y_2(x) = xe^{-x}, y_3(x) = x^2e^{-x}$. These three solutions are linearly independent (you don't have to verify that these are solutions, or that they are linearly independent).

Write down the general solution of this differential equation.

Solution: By the principle of superposition, the general solution is

$$y(x) = C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x}.$$

2. (15 points) Calculate a particular solution of

$$y'' + 4y' + 4y = 2e^x.$$

Solution: We make a guess $y(x) = Ae^x$, for a constant A.
Since $y''(x) = y'(x) = y(x) = Ae^x$, we have

$$(A + 4A + 4A)e^x = 2e^x,$$

which implies $9A = 2$, and so $A = 2/9$. Our particular solution is thus $y = 2e^x/9$.

3. (15 points) Consider a mass-spring system with a weight of mass 2kg. The spring is such that if you pull on it with a force of 0.5 Newtons, the spring stretches 0.1 meters. There is no external force acting on the weight, and the effect of friction is negligible. What is the period (in seconds) of the spring's oscillation? You are not allowed to use the formulas for frequency and period in your book without justification.

Solution: Since 0.5 N stretches the spring 0.1 meters, 5N of force will stretch the spring 1 meter. The spring constant is thus $k = 5$.

This means that our equation is

$$2x''(t) + 5x(t) = 0.$$

Substituting $x(t) = e^{rt}$, we obtain

$$2r^2 + 5 = 0,$$

or $r = \pm\sqrt{5/2}i$. This implies a general solution of

$$x(t) = C_1 \cos(\sqrt{5/2}t) + C_2 \sin(\sqrt{5/2}t).$$

To calculate the period T, we need to write

$$\sqrt{5/2}T = 2\pi,$$

which implies a period of $T = \frac{2\pi}{\sqrt{5/2}}$ seconds.

4. (15 points) A spring system has a weight of mass 1kg, damping constant 6Ns/m, and spring constant 45 N/m. There is an external force of $F_0 \cos(\omega t)$ acting on the weight. Is there practical resonance in this system? If there is, calculate the practical resonance frequency (in radians/second). You may use the following formula for the amplitude of a forced oscillation:

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}.$$

Solution: We seek to maximize

$$C(\omega) = \frac{F_0}{\sqrt{(45 - \omega^2)^2 + (6\omega)^2}}.$$

This is the same as minimizing the expression under the square root. So let us define

$$D(\omega) = (45 - \omega^2)^2 + (6\omega)^2.$$

Taking the derivative, we have

$$D'(\omega) = 2(45 - \omega^2)(-2\omega) + 72\omega.$$

Or simplifying,

$$D'(\omega) = 4\omega^3 - 108\omega = 4\omega(\omega^2 - 27)$$

So setting $D'(\omega) = 0$ gives us solutions $\omega = 0, \omega = \sqrt{27}, \omega = -\sqrt{27}$. The only answer that makes sense is $\omega = \sqrt{27}$.

5. (15 points) Consider the boundary value problem given by

$$y'' + \lambda y = 0, y(0) = 0, y'(10) = 0.$$

(a) Is $\lambda = 0$ an eigenvalue? If it is, write down a corresponding eigenfunction.

(b) Identify all the negative eigenvalues, if there are any. You **do not** need to find the corresponding eigenfunctions.

Solution: We consider the case $\lambda = 0$. Our equation becomes $y'' = 0$, which means we have the general solution $y(x) = Ax + B$ for constants A, B. We have $y'(x) = A$, so we have $A = 0$, and $10A + B = 0$. The only solution for this algebraic system is $A = B = 0$, so we conclude that $y(x) = 0$ and so $\lambda = 0$ is not an eigenvalue.
We consider now the case $\lambda < 0$. We write $\lambda = -\alpha^2$ for notational reasons. We then have

$$y'' - \alpha^2 y = 0.$$

Making the substitution $y = e^{rx}$ we get

$$r^2 - \alpha^2 = 0,$$

which has roots $r = \pm\alpha$. Our general solution is thus

$$y(x) = C_1 \exp(\alpha x) + C_2 \exp(-\alpha x).$$

Taking the derivative, we have

$$y'(x) = C_1 \alpha \exp(\alpha x) - C_2 \alpha \exp(-\alpha x).$$

Plugging in the boundary condition $y(0) = 0$ we have

$$0 = C_1 + C_2,$$

and so $C_2 = -C_1$. Plugging in the boundary condtion $y'(10) = 0$ wehave

$$0 = C_1 \alpha \exp(10\alpha) - C_2 \alpha \exp(-10\alpha).$$

Since $C_2 = -C_1$, this simplifies to

$$0 = \alpha C_1 (\exp(10\alpha) + \exp(-10\alpha)).$$

Note that $\alpha > 0$, and also that $\exp(10\alpha)$ and $\exp(-10\alpha)$ are both strictly positive. Thus we must have $C_1 = 0$, which implies $C_2 = 0$, and so $y(x) = 0$. We conclude that there are no negative eigenvalues.

6. Find the general solution of the system

$$x'(t) = -y(t), y'(t) = 13x(t) + 4y(t).$$

Solution: We differentiate the second equation to obtain

$$y''(t) = 13x'(t) + 4y'(t).$$

Plugging in the first equation we get

$$y''(t) = 13(-y(t)) + 4y'(t).$$

Once we rearrange terms, this gets us

$$y''(t) - 4y'(t) + 13y(t) = 0.$$

Setting $y(t) = e^{rt}$ we get that

$$r^2 - 4r + 13 = 0,$$

and so

$$r = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i.$$

Our general solution for $y(t)$ is

$$y(t) = e^{2t}(C_1 \cos(3t) + C_2 \sin(3t)).$$

We then have, by the product rule,

$$y'(t) = 2e^{2t}(C_1 \cos(3t) + C_2 \sin(3t)) + e^{2t}(-3C_1 \sin(3t) + 3C_2 \cos(3t)).$$

So our general solution for $x(t)$ is

$$\begin{aligned} x(t) =& \frac{y'(t) - 4y(t)}{13} \\ =& \frac{1}{13}[2e^{2t}(C_1 \cos(3t) + C_2 \sin(3t)) + e^{2t}(-3C_1 \sin(3t) + 3C_2 \cos(3t)) \\ & - 4(e^{2t}(C_1 \cos(3t) + C_2 \sin(3t)))] \end{aligned}$$

7. (10 points) Given a solution $y(x)$ of a linear homogeneous equation

$$y'' + p(x)y'(x) + q(x)y(x) = 0,$$

we know that $Cy(x)$ is also a solution for any constant C.

Is this fact still true for a nonhomogeneous equation,

$$y'' + p(x)y'(x) + q(x)y(x) = f(x),$$

where $f(x)$ is not necessarily zero? If it is true, prove it. If it is not true, demonstrate a counterexample.

Solution: The "fact" is false for nonhomogeneous solutions. A simple example is $y''(x) = 1$. $y(x) = x^2/2$ is a solution to this equation, but $y(x) = -x^2/2$ is not a solution.