MATH 3113-009 Test II

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March 25, 2015 10:30am-11:20am

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. No calculators allowed.

Name: ____________________

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

1. (20 points) Let $p(x), q(x), r(x), f(x)$ be continuous functions.

   The following differential equation,

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0$$

   has solutions $y_1 = \sin(x), y_2 = e^{-2x}, y_3 = \cos(x)$. These three functions are linearly independent (you DON'T have to prove this).

   We also know that

$$\sinh(x)''' + p(x)\sinh(x)'' + q(x)\sinh(x)' + r(x)\sinh(x) = f(x).$$

   Write down the general solution for

$$y''' + p(x)y'' + q(x)y' + r(x)y = f(x).$$

Solution: By the principle of superposition, the complementary solution is

$$y_c = C_1\sin(x) + C_2e^{-2x} + C_3\cos(x).$$

We are given that the particular solution is $y_p = \sinh(x)$.

Since the general solution is $y = y_c + y_p$, we have

$$y = C_1\sin(x) + C_2e^{-2x} + C_3\cos(x) + \sinh(x).$$

2. (20 points) Calculate the general solution for the differential equation

$$y'' + 4y' + 6y = 0.$$

Solution: We substitute $y = e^{rx}$ to obtain

$$r^2 + 4r + 6 = 0.$$

By the quadratic formula, we find roots $r = -2 \pm i\sqrt{2}$. This corresponds to linearly independent solutions

$$y_1 = e^{-2x}\cos(\sqrt{2}x), y_2 = e^{-2x}\sin(\sqrt{2}x).$$

Our general solution is thus

$$y = C_1 e^{-2x}\cos(\sqrt{2}x) + C_2 e^{-2x}\sin(\sqrt{2}x).$$

3. (15 points) Calculate the particular solution of

$$y'' + 3y' + 2y = 4e^x,$$

using the variation of parameters formula,

$$y_p(x) = -y_1(x) \int \frac{y_2(x) f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)} dx,$$

where $W(x)$ is the Wronskian of y_1 and y_2, and y_1, y_2 are two linearly independent solutions of the homogeneous equation.

For partial credit, you may use the principle of undetermined coefficients to solve this problem instead.

Solution: First, we solve the homogeneous equation

$$y'' + 3y' + 2y = 0.$$

We substitute $y = e^{rx}$ and we get $r^2 + 3r + 2 = 0$, which has roots $r = -1, r = -2$. Thus we have $y_1 = e^{-x}, y_2 = e^{-2x}$.

The Wronskian of y_1, y_2 is

$$W(x) = e^{-x}(2e^{-2x}) - e^{-2x}e^{-x} = e^{-3x}.$$

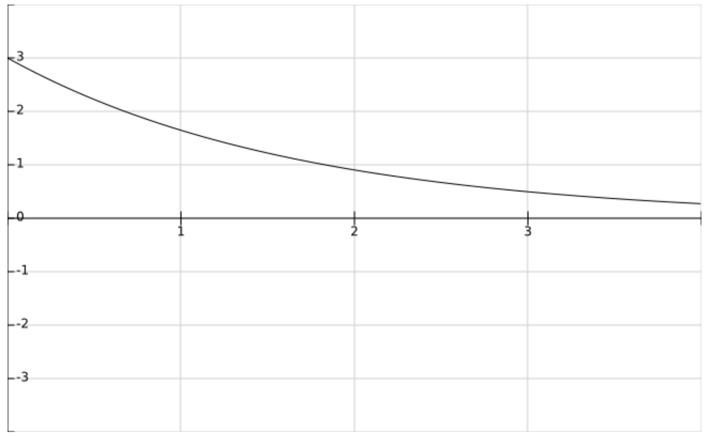
The nonhomogeneous part is $f(x) = 4e^x$. We can then calculate

$$\begin{aligned} \int \frac{y_2(x) f(x)}{W(x)} dx &= \int \frac{e^{-2x}(4e^x)}{e^{-3x}} dx \\ &= 2e^{2x}. \\ \int \frac{y_1(x) f(x)}{W(x)} dx &= \int \frac{e^{-x}(4e^x)}{e^{-3x}} dx \\ &= \frac{4}{3}e^{3x}. \end{aligned}$$

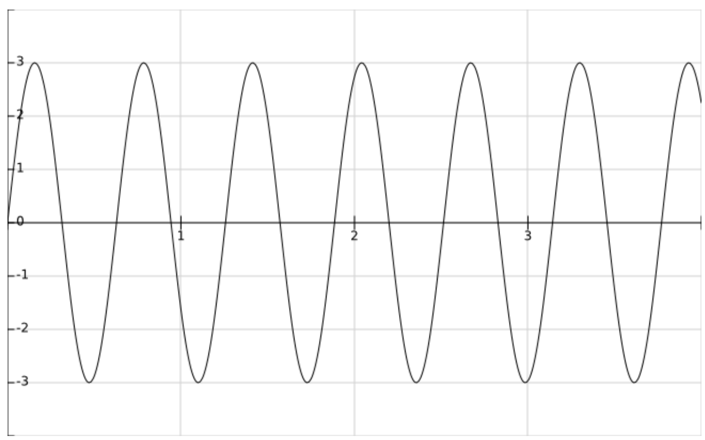
Plugging in all our calculations in the formula, we have

$$y_p = \frac{2}{3}e^x.$$

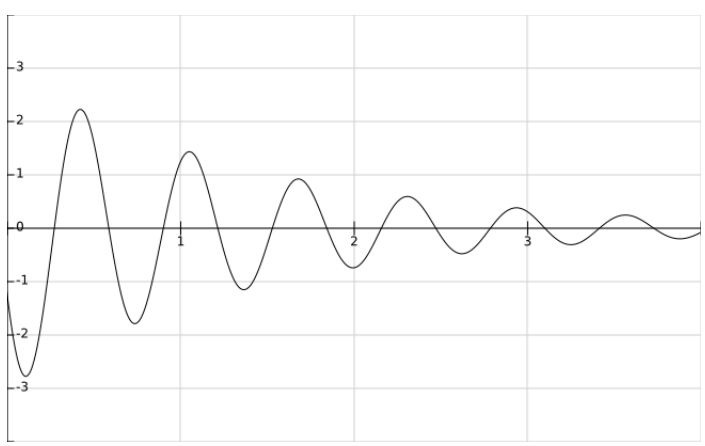
i)



ii)



iii)



4. (18 points) You are given the following values for mass m, damping constant c, and spring constant k, corresponding to a mass-spring system with no external force. Indicate whether each system is undamped, underdamped, or overdamped. Also, indicate which of the three graph plots on the previous page most closely matches the graph of $x(t)$, the position of the weight at time t (Ignore the numbers on the graph axes).

   a) $m = 3, c = 0, k = 2$

   b) $m = 2, c = 3, k = 1$

   c) $m = 2, c = 3, k = 2$

Solution:

a) $m = 3, c = 0, k = 2$ Undamped, graph (ii).

b) $m = 2, c = 3, k = 1$ Overdamped, graph (i).

c) $m = 2, c = 3, k = 2$ Underdamped, graph (iii).

5. (17 points) Consider the boundary value problem given by

$$y'' + \lambda y = 0, y'(0) = 0, y'(\pi) = 0.$$

(a) Is $\lambda = 0$ an eigenvalue? If it is, write down a corresponding eigenfunction.

(b) Identify all the positive eigenvalues. You **do not** need to find the corresponding eigenfunctions.

Solution: We consider the case $\lambda = 0$. Our equation becomes $y'' = 0$, which means we have the general solution $y(x) = Ax + B$ for constants A, B. We have $y'(x) = A$, so we have $A = 0$, and no restrictions on B. We thus have solutions $y = B$ for arbitrary constants B. $\lambda = 0$ is thus an eigenvalue with eigenfunction $y = 1$.

We consider now the case $\lambda > 0$. We write $\lambda = \alpha^2$ for notational reasons. We then have

$$y'' + \alpha^2 y = 0.$$

Making the substitution $y = e^{rx}$ we get

$$r^2 + \alpha^2 = 0,$$

which has roots $r = \pm i\alpha$. Our general solution is thus

$$y(x) = C_1 \cos(\alpha x) + C_2 \sin(\alpha x).$$

Taking the derivative, we have

$$y'(x) = -C_1 \alpha \sin(\alpha x) + C_2 \alpha \cos(\alpha x).$$

Plugging in the boundary condition $y'(0) = 0$ we have

$$0 = C_2 \alpha,$$

and so $C_2 = 0$. Plugging in the boundary condtion $y'(\pi) = 0$ and using $C_2 = 0$ we have

$$0 = C_1 \alpha \sin(\alpha \pi).$$

Note that $\alpha > 0$. So C_1 can be nonzero only when $\sin(\alpha\pi) = 0$, which happens when $\alpha = 1, 2, 3, 4, \ldots$. Since $\lambda = \alpha^2$, our eigenvalues are

$$\lambda = 1, 4, 9, 16, \ldots.$$

6. (10 points) Let $q(x), p(x)$ be continuous functions on the real line. It is known that the general solution of a second order linear homogeneous differential equation,

$$y''(x) + q(x)y'(x) + p(x)y(x) = 0 \tag{1}$$

has a general solution that contains two arbitrary constants. Consider the following statement:

*If $y_1(x), y_2(x)$ are **linearly independent** solutions to equation* (1)*, then the general solution of* (1) *can be given as*

$$y(x) = C_1 y_1(x) + C_2 y_2(x),$$

where C_1, C_2 are the two arbitrary constants.

Explain why the *linearly independent* condition is necessary.

Solution: If $y_1(x), y_2(x)$ are linearly dependent, we have $y_1(x)/y_2(x) = K$ for some constant K. We can then rewrite

$$y(x) = C_1 K y_2(x) + C_2 y_2(x) = (C_1 K + C_2) y_2(x) = C_3 y_2(x),$$

where $C_3 = C_1 K + C_2$. But we have rewritten the general solution of a second order equation so that it has only one arbitrary constant, which is impossible.