MATH 3113-008 Test II

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Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. No calculators allowed.

Name: ____________________

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

1. (20 points) Demonstrate that the functions $\{\cos(x), \sin(x), e^x\}$ are linearly independent. For partial credit, you may instead demonstrate that two of the three functions are linearly independent (You may choose which two).

Solution: We set up the Wronskian determinant

$$W(x) = \det \begin{pmatrix} \cos(x), \sin(x), e^x \\ -\sin(x), \cos(x), e^x \\ -\cos(x), -\sin(x), e^x \end{pmatrix}.$$

We calculate that this is equal to

$$\begin{aligned} W(x) =& \cos(x)(\cos(x)e^x + e^x \sin(x)) - \sin(x)(-\sin(x)e^x + e^x \cos(x)) + e^x(\sin^2(x) + \cos^2(x)). \\ =& 2e^x(\cos^2(x) + \sin^2(x)) \\ =& 2e^x \neq 0. \end{aligned}$$

Thus the three functions are linearly independent.

2. (20 points) Calculate the general solution for the differential equation

$$y''' + 3y'' + 3y' + y = 0.$$

You may assume that any finite subset of the functions

$$\{x^a e^{bx}\cos(cx), x^a e^{bx}\sin(cx) \mid a, b, c, \text{real numbers}\}$$

is linearly independent as long as the subset does not contain the same function twice. Hint: $(r+1)^3 = r^3 + 3r^2 + 3r + 1$.

Solution: We use the substitution $y = e^{rx}$. We then get $r^3+3r^2+3r+1 = 0$. Using the hint, we determine that the equation has a triple root $r = -1, -1, -1$.

The general solution is then

$$y = C_1 e^{-x} + C_2 x e^{-x} + C_3 x^2 e^{-x}.$$

3. (15 points) Use the method of undetermined coefficients to calculate a particular solution of

$$y'' - 3y' + 3y = 6x^2 - 3x - 8.$$

Solution: We make the guess $y = Ax^2 + Bx + C$. We then have $y' = 2Ax + B$, $y'' = 2A$. Substituting that in, we have

$$2A - 3(2Ax + B) + 3(Ax^2 + Bx + C) = 6x^2 - 3x - 8.$$

We match the x^2 coefficients to get $3A = 6$, and so $A = 2$. We match the x coefficients to get $-6A + 3B = -3$, or $B = 3$. We match the constant coefficients and we get

$$2A - 3B + 3C = -8,$$

which gives us $C = -1$. Thus our particular solution is

$$y_p = 2x^2 + 3x - 1.$$

4. (18 points) a) A weight of mass 5kg is suspended on a spring. It is known that whenever a force of 2 Newtons is applied to the spring, the spring stretches by 0.5 meters. The effect of friction on this mass-spring system is negligible. A force of $3\cos(\omega t)$ Newtons is applied to the mass-spring system. For what value of ω will the external force cause resonance?

   b) Consider instead a different mass-spring system with mass $m = 1$, $c = 2, k = 3$, with an external force of $F_0 \cos(\omega t)$. Is there a value of ω that will cause practical resonance? If there is, identify the practical resonance frequency. Recall that the amplitude of a system experiencing external force of angular frequency ω is

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}.$$

Solution:

(a) We need 4 Newtons to stretch the spring 1 meter, so $k = 4$. To find the natrual frequency of the system, we solve

$$5x''(t) + 4x(t) = 0.$$

Substituting $x(t) = e^{rt}$. We then get $5r^2 + 4 = 0$, or $r = i\sqrt{4/5}$. The general solution is

$$x(t) = C_1 \cos(\sqrt{4/5}t) + C_2 \sin(\sqrt{4/5}t).$$

The natural (angular) frequency is $\sqrt{4/5}$. Thus we have resonance when

$$\omega = \sqrt{4/5}.$$

(b) We want to maximize $C(\omega)$, which is equivalent to minimizing the denominator

$$D(\omega) = (3 - \omega^2)^2 + (2\omega)^2.$$

Differentiating, we obtain

$$D'(\omega) = 2(3 - \omega^2)(-2\omega) + 8\omega = -4\omega + 4\omega^3 = 4\omega(\omega + 1)(\omega - 1).$$

Since frequency must be positive, we thus have a minimum when $\omega = 1$. This is the practical resonance frequency.

5. (17 points)

(a) Write down

$$x^{(3)} + 3x'' + 2x' - 5x = \sin(2t)$$

as a system of three first-order differential equations.

(b) Find the general solution of the system of equations

$$\begin{aligned} x'(t) =& y(t), \\ y'(t) =& 6x(t) - y(t). \end{aligned}$$

Solution:

(a) We write $x_1 = x$, $x_2 = x'$, $x_3 = x''$. We can then rewrite the equation as

$$x_1' = x_2, x_2' = x_3, x_3' = \sin(2t) + 5x_1 - 2x_2 - 3x_3.$$

(b) We differentiate the second equation to obtain $y''(t) = 6x'(t) - y'(t)$. We substitute $x'(t) = y(t)$ and obtain the equation $y''(t) + y'(t) - 6y(t) = 0$. We substitute $y = e^{rt}$ to obtain $r^2 + r - 6 = 0$, which has roots $r = 3, r = -2$. Our general solution for y is thus

$$y(t) = C_1 e^{3t} + C_2 e^{-2t}.$$

We then differentiate to find

$$y'(t) = 3C_2 e^{3t} - 2C_2 e^{-2t}.$$

The second given equation tellls us that

$$x(t) = \frac{y'(t) + y(t)}{6} = \frac{3C_2 e^{3t} - 2C_2 e^{-2t} + C_1 e^{3t} + C_2 e^{-2t}}{6}.$$

We thus have general solutions for both $x(t)$ and $y(t)$.

6. (10 points) The continuous functions $p(x)$, $q(x)$, $f(x)$, and $g(x)$ are given. It is known that

$y(x) = \cos(x)\cosh(x)$ is a solution for

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x),$$

and that $y(x) = x^3 e^x$ is a solution for

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x).$$

Find a particular solution for the differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 3f(x) - \pi g(x).$$

Solution: We just take $y = 3\cos(x)\cosh(x) - \pi x^3 e^x$, and distribute derivatives. Since

$$3[\cos(x)\cosh(x)]'' + p(x)3[\cos(x)\cosh(x)]' + q(x)3[\cos(x)\cosh(x)] = 3f(x),$$

and

$$\pi[x^3e^x]'' + p(x)\pi[x^3e^x]' + q(x)\pi[x^3e^x] = \pi g(x),$$

subtracting the two equations above gets us

$$[3\cos(x)\cosh(x) - \pi x^3e^x]'' + p(x)[3\cos(x)\cosh(x) - \pi x^3e^x]' + q(x)[3\cos(x)\cosh(x) - \pi x^3e^x]$$
$$= 3f(x) - \pi g(x).$$