MATH 4163-002 Test II

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March 10, 2016 1:30pm-2:45pm

Answer the questions in the spaces provided on the question sheets. No calculators allowed.

Name: ____

Problem	Score
1	
2	
3	
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Total	

1. (10 points) We say that an operator L is self-adjoint if for any two functions u, v and a multiplication $*$ on functions, $L(u) * v = u * L(v)$. Write down two different examples of self-adjoint operators, if we let * be standard multiplication.

Solution: $L(f) = 0$ and $L(f) = f$ are the simplest examples.

2. (15 points) Recall that a Fourier series on $[-L, L]$ can be written down in two forms: either as

$$\hat{f}(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L},$$

or as

$$\hat{f}(x) = \sum_{m=-\infty}^{\infty} C_m e^{-im\pi x/L}.$$

Express C_m in terms of the A_n and B_n (recall that the m in C_m can be positive or negative, but the n in A_n, B_n cannot be negative.) You may use the trigonometric identities $2\cos(\theta) = e^{i\theta} + e^{-i\theta}$, and $2i\sin(\theta) = e^{i\theta} - e^{-i\theta}$.

Solution: We plug in the hint to the first version of the Fourier series.

$$\hat{f}(x) = A_0 + \sum_{n=1}^{\infty} A_n \frac{1}{2}\left(\exp\frac{n\pi x}{L} + \exp\frac{-n\pi x}{L}\right) + B_n \frac{1}{2i}\left(\exp\frac{n\pi x}{L} - \exp\frac{-n\pi x}{L}\right),$$

Collecting like terms, we can rewrite this as

$$\hat{f}(x) = A_0 + \sum_{n=1}^{\infty} \frac{A_n + B_n/i}{2}\exp\frac{n\pi x}{L} + \frac{A_n - B_n/i}{2}\exp\frac{-n\pi x}{L}.$$

So clearly, if m is positive,

$$C_m = \frac{A_n - B_n/i}{2}, C_{-m} = \frac{A_n + B_n/i}{2},$$

and

$$C_0 = A_0.$$

3. (15 points) Consider a vibrating string lying between x and $x + \Delta x$, and let $u(x,t)$ be the height of the string at time t and location x. The vertical acceleration of the string applies a force of $\rho \Delta x \frac{\partial^2}{\partial t^2} u(x,t)$. The tension of the string applies a vertical force of $T\sin(\theta(x+\Delta x, t)) - T\sin(\theta(x,t))$, where T is the (constant) tension of the string, and $\theta(x,t)$ is the angle of the string's slope at position x and time t. Also, you may assume that $\theta(x,t)$ is always small enough that $\sin(\theta(x,t))$ is always roughly equal to $\tan(\theta(x,t)$.

Derive the vibrating string equation from this information (you may want to start by equating all the vertical forces acting on the string)

Solution: We have

$$\rho \Delta x \frac{\partial^2}{\partial t^2} u(x,t) = T\sin(\theta(x+\Delta x, t)) - T\sin(\theta(x,t)).$$

Let us divide by Δx to obtain

$$\rho \frac{\partial^2}{\partial t^2} u(x,t) = \frac{T\sin(\theta(x+\Delta x, t)) - T\sin(\theta(x,t))}{\Delta x}.$$

Taking the limit as $\Delta x \to 0$ we have

$$\rho \frac{\partial^2}{\partial t^2} u(x,t) = \frac{d}{dx} T\sin(\theta(x,t)).$$

However, notice that the slope of $u(x,t)$ can be expressed in two ways: as $\frac{\partial}{\partial x} u(x,t)$, and as rise over run. Rise over run is equivalent to opposite over adjacent, which is equivalent to $\tan(\theta(x,t)$, and we can assume that $\tan(\theta(x,t) \sim \sin(\theta(x,t))$. Thus we may assume

$$\frac{\partial}{\partial x} u(x,t) = \sin(\theta(x,t)).$$

Plugging this in to our previous equation, we get

$$\rho \frac{\partial^2}{\partial t^2} u(x,t) = T \frac{\partial^2}{\partial x^2} u(x,t),$$

which is the vibrating string equation.

4. (15 points) Product solutions to the vibrating string equation take the form

$$u_n(x,t) = \sin\frac{n\pi x}{L}\left(A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right), n = 1, 2, 3, \ldots$$

Explan what happens to the pitch of the sound produced by the string's vibration if we

(a) increase the length of the string

(b) increase the density of the string

(c) increase the tension of the string

Justify your answers using the above formula for the product solution. Justifications of the form "I know this is true because I play guitar" will earn very little partial credit.

Solution: The frequency of the trig functions $\sin(\omega t), \cos(\omega t)$ is $\frac{\omega}{2\pi}$ hertz. Thus if we increase L, the ω decreases, and the frequency of the vibration decreases, resulting in a lower pitch.

$c = \sqrt{\frac{T}{\rho}}$, and so increasing tension increaes c and therefore ω, leading to a higher pitch. Conversely, increaseing the density lowers the T and therefore ω, thus decreasing the pitch.

5. (15 points) Consider the Sturm-Liouville operator,

$$L(F(x)) = \frac{d}{dx}\left(p(x)\frac{dF(x)}{dx}\right) + q(x)F(x).$$

You may recall the Green's formula,

$$\int_a^b (uL(v) - vL(u))dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_{x=a}^{b}.$$

Show that if we impose boundary conditions $3f(a) - f'(a) = 0, f'(b) = 0$ on both $f = u$ and $f = v$, the operator L is self-adjoint.

Solution: Self-adjointness is equivalent to showing

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_{x=a}^{b} = 0.$$

We expand out the LHS to get

$$p(b)\left(u(b)\frac{dv(b)}{dx} - v(b)\frac{du(b)}{dx}\right) - p(a)\left(u(a)\frac{dv(a)}{dx} - v(a)\frac{du(a)}{dx}\right).$$

Since $u'(b) = v'(b) = 0$, the entire first term is zero. This leaves us with

$$p(a)\left(u(a)\frac{dv(a)}{dx} - v(a)\frac{du(a)}{dx}\right).$$

But note that since $3u(a) - u'(a)$ and $3v(a) - v'(b)$ are both zero, we must have $u'(a) = 3u(a)$ and $v'(a) = 3v(a)$. This implies

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_{x=a}^{b} = p(a)\left(u(a)3v(a) - v(a)3u(a)\right) = 0.$$

6. (15 points) Let λ_m, λ_n be two eigenvalues for a self-adjoint Sturm-Liouville equation

$$L(F(x)) + \lambda\sigma(x)F(x) = 0,$$

and let $F_m(x), F_n(x)$ respectively be their two eigenfunctions.
Show that if $\lambda_m \neq \lambda_n$, then

$$\int_a^b F_m(x)F_n(x)\sigma(x)dx = 0.$$

Hint: what does the fact that L is self-adjoint imply?

Solution:

Using self-adjointness, we know that

$$\int_a^b L(F_m)F_n - L(F_n)F_m dx = 0.$$

But we also know from the Sturm-Liouville equation that

$$L(F_m(x)) = -\lambda_m\sigma(x)F_m(x),$$

and

$$L(F_n(x)) = -\lambda_n\sigma(x)F_n(x).$$

Plugging these into the self-adjointness equation we get

$$\int_a^b -\lambda_m\sigma(x)F_m(x)F_n(x) + \lambda_n\sigma(x)F_n(x)F_m(x)dx = 0$$

Factoring, this becomes

$$(\lambda_n - \lambda_m)\int_a^b F_m(x)F_n(x)\sigma(x)dx = 0,$$

but since $\lambda_n - \lambda_m \neq 0$, it must be true that

$$\int_a^b F_m(x)F_n(x)\sigma(x)dx = 0.$$

7. (15 points) It is known that if our Sturm-Liouville operator L is self-adjoint and if we impose Dirichlet, Neumann, or Robin boundary conditions, then where $F_1(x), F_2(x)$ are two eigenfunctions corresponding to the same eigenvalue λ,

$$F_1(x)F_2'(x) - F_2(x)F_1'(x) = 0.$$

Using this, show that $F_2(x)$ must be a constant multiple of $F_1(x)$.

Solution: By the quotient rule,

$$\frac{d}{dx}\left(\frac{F_2(x)}{F_1(x)}\right) = \frac{F_1(x)F_2'(x) - F_2(x)F_1'(x)}{F_1(x)^2} = 0$$

This implies that for some constant C,

$$\frac{F_2(x)}{F_1(x)} = C,$$

or $F_2(x) = CF_1(x)$.