MATH 3113-002 Test II

Dr. Darren Ong

October 28, 2015 11:30am-12:20pm

Answer the questions in the spaces provided on the question sheets. No calculators allowed.

Name: ____________________

Answers

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

1. (20 points) Calculate the general solution for the differential equation

$$y''(x) + 4y'(x) + 29y = 0.$$

Solution:

$y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

$r^2 e^{rx} + 4re^{rx} + 29e^{rx} = 0$

$e^{rx}(r^2 + 4r + 29) = 0$

$r^2 + 4r + 29 = 0$

$r = \frac{-4 \pm \sqrt{16 - 4 \cdot 29}}{2}$

$= \frac{-4 \pm \sqrt{-100}}{2}$

$= -2 \pm 5i$

This implies

$$\boxed{y = C_1 e^{-2x}\cos(5x) + C_2 e^{-2x}\sin(5x)}$$

is the general solution

2. (20 points) Find the general solution for

$$y''' + 7y'' + 8y' - 16y = 0.$$

Solution:

$y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

$y''' = r^3 e^{rx}$

$$(r^3 + 7r^2 + 8r - 16)e^{rx} = 0$$

$$r^3 + 7r^2 + 8r - 16 = 0$$

guess that $r = 1$ is a root

$$(1)^3 + 7(1)^2 + 8(1) - 16 = 0$$

$\Rightarrow$ $r-1$ is a factor

$$\begin{array}{r|l} & r^2 + 8r + 16 \\ \hline r-1 & r^3 + 7r^2 + 8r - 16 \\ & r^3 - r^2 \\ \hline & \quad 8r^2 + 8r \\ & \quad 8r^2 - 8r \\ \hline & \qquad 16r - 16 \\ & \qquad 16r - 16 \\ \hline & \qquad\quad 0 \end{array}$$

So $\quad r^3 + 7r^2 + 8r - 16 = (r-1)(r^2 + 8r + 16)$

$$= (r-1)(r+4)^2$$

roots are $r = 1, -4, -4$

general solution is $\boxed{y = C_1 + C_2 e^{-4x} + C_3 x e^{-4x}}$

3. (10 points) Rewrite

$$x^{(4)}(t) - 7x'''(t) + 2x'(t) + x(t) = e^t$$

as a system of four first-order differential equations.

Solution:

Define

$$x_1 = x$$
$$x_2 = x'$$
$$x_3 = x''$$
$$x_4 = x'''$$

We have $x_1' = x_2$, $x_2' = x_3$, $x_3' = x_4$

$$x_4' = x^{(4)} = 7x''' - 2x' - x + e^t$$
$$= 7x_4 - 2x_2 - x_1 + e^t$$

So our system is

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ x_4' &= 7x_4 - 2x_2 - x_1 + e^t \end{aligned}}$$

4. (20 points) Consider a mass-spring system with a weight of $1kg$, a damping constant of $3Ns/m$ and a spring constant of $2N/m$. There is also an external force of $10\cos(t)$ Newtons acting on the weight at time t. At the beginning of the experiment, the weight is at the $2m$ position with no initial velocity. Calculate the transient and steady periodic solutions of this system.

Solution:

$m=1, \quad c=3, \quad k=2$

$$\boxed{x''(t)+3x'(t)+2x(t) = 10\cos(t)}$$

Find $\cancel{x}_p$

$\cancel{x}_p = A\cos(t)+B\sin(t)$

$x_p' = -A\sin(t)+B\cos(t)$

$x_p'' = -A\cos(t)-B\sin(t)$

$(-A\cos(t)-B\sin(t))+3(-A\sin(t)+B\cos(t))$
$+2(A\cos(t)+B\sin(t)) = 10\cos(t)$

Match cos coefficients: $A+3B=10$

Match sin coefficients: $B-3A=0$

$\therefore A=1, B=3$

$$\boxed{x_p = \cos(t)+3\sin(t)}$$

Find x_c Homogeneous part is $x''(t)+3x'(t)+2x(t)=0$

$x=e^{rt}$
$x'=re^{rt}$
$x''=r^2e^{rt}$

$r^2+3r+2=0$
$(r+2)(r+1)=0$

$r=-1,-2$

$$\boxed{x_c = c_1e^{-2t}+c_2e^{-t}}$$

General solution is ~~xxx~~ $x_p+x_c = c_1e^{-2t}+c_2e^{-t}+\cos(t)+3\sin(t)$

$x(t) = c_1e^{-2t}+c_2e^{-t}+\cos(t)+3\sin(t)$

Initial conditions: $x(0)=2, \quad x'(0)=0$

$x'(t) = -2c_1e^{-2t}-c_2e^{-t}-\sin(t)+3\cos(t)$

$x(0)=2 \Rightarrow 2 = c_1+c_2+1$

$x'(0)=0 \Rightarrow 0 = -2c_1-c_2+3$

$\therefore$ ~~xxx~~ $c_1=2, \quad c_2=-1$

$$\boxed{x(t) = 2e^{-2t}-e^{-t}+\cos(t)+3\sin(t)}$$

~~particular~~ IVP solution

$\boxed{\text{Transient solution is } x_{tr} = 2e^{-2t}-e^{-t}}$

$\boxed{\text{Steady periodic solution is } x_{sp} = \cos(t)+3\sin(t)}$

5. (20 points) Find the **positive** eigenvalues for the boundary value problem

$$y'' + \lambda y = 0,$$

with boundary conditions $y(0) = 0$ and $y'(2\pi) = 0$. Also, write down an eigenfunction corresponding to the smallest positive eigenvalue.

Solution:

$\lambda = \alpha^2, \quad \alpha > 0$

$y'' + \alpha^2 y = 0$

$y = e^{rx}$

$y' = re^{rx}$ $\quad r^2 + \alpha^2 = 0$

$y'' = r^2 e^{rx}$ $\quad r^2 = -\alpha^2$

$r = \pm \alpha$

$y(x) = C_1 \cos(\alpha x) + C_2 \sin(\alpha x)$

$y(0) = 0 \Rightarrow \quad 0 = C_1$

$y'(x) = \alpha C_2 \cos(\alpha x)$

$y'(2\pi) = 0 \Rightarrow \quad \cancel{0 = \alpha C_2 (2\pi\alpha)}$

$0 = \alpha C_2 \cos(2\pi\alpha)$

For $C_2 \neq 0$, we need $\cos(2\pi\alpha) = 0$.

This happens when $\alpha = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \ldots$

$\alpha = \frac{n}{4}$, n odd

Eigenvalues are $\boxed{\lambda = \frac{n^2}{16}, \quad n \text{ odd.}}$

Smallest eigenvalue is $\frac{1}{16}$.

$\alpha = \frac{1}{4}$, So eigenfunction is $\boxed{y(x) = \sin\left(\frac{x}{16}\right).}$

$C_2 = 0$

6. (10 points) Gideon wants to find the general solution of the differential equation

$$xy''(x) + y'(x) + 2 = 0.$$

Gideon notices that this equation is linear since there are no squares of y in the equation, and homogeneous since the right hand side is zero. He thus decides to use the principle of superposition.

Gideon finds that $y_1(x) = -2x$ and $y_2(x) = 1 - 2x$ are solutions. He checks that $y_1(x)$ and $y_2(x)$ are linearly independent, and so he concludes that

$$y(x) = C_1(-2x) + C_2(1 - 2x)$$

is the general solution.

Gideon's friend Kiora tries $C_1 = 1, C_2 = -1$ to obtain a particular solution $y(x) = -1$. However, when she plugs in this equation in the general solution, it doesn't work. Kiora tells Gideon that his general solution is wrong. What was Gideon's mistake?

For partial credit, you may instead write down the defnitions of "linear", "homogeneous" and the principle of superposition.

The equation is not homogeneous
since 2 is a non-homogeneous term
(it does not contain y)