MATH 3113-001 Test II

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October 28, 2015 1:30pm-2:20pm

Answer the questions in the spaces provided on the question sheets. No calculators allowed.

Name: ______________________________

Answers

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

1. (20 points) Calculate the general solution for the differential equation

$$y'' - 8y' + 20y = 0$$

Solution:

$y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

$r^2 e^{rx} - 8re^{rx} + 20e^{rx} = 0$

$e^{rx}(r^2 - 8r + 20) = 0$

$r^2 - 8r + 20 = 0$

$r = \dfrac{8 \pm \sqrt{64 - 80}}{2}$

$= 4 \pm 2i$

General solution is $\boxed{y(x) = C_1 e^{4x} \cos(2x) + C_2 e^{4x} \sin(2x)}$

2. (10 points) A mass-spring system has a weight of mass $2kg$, and the spring stretches 1.5 meters when a force of 27 Newtons stretches it. We apply an external force of $3\cos(\omega t)$ Newtons at time t seconds. We ignore the effect of friction. For what frequency ω does resonance occur?

Solution:

resonance frequency = natural frequency.

$k = \frac{27}{1.5} = 18$

$2x'' + 18x = 0$

$x = e^{rt}$

$x'' = r^2 e^{rt}$

$2r^2 + 18 = 0$

$r^2 = -9$

$r = \pm 3i$

general solution

$x(t) = C_1 \cos(3t) + C_2 \sin(3t)$

natural frequency = 3.

resonance frequency = 3.

3. (20 points) Solve the following initial value problem, where x and y are functions of t.

$$\begin{aligned} x' &= \frac{1}{2}y \\ y' &= -2x + 2y \\ x(0) &= 2 \\ y(0) &= -4 \end{aligned}$$

Solution:

$x'' = \frac{1}{2} y'$

$2x'' = -2x + 2y$ (since $y = 2x'$)

$2x'' = -2x + 4x'$

$2x'' - 4x' + 2x = 0$

$x = e^{rt}$

$x' = re^{rt}$

$x'' = r^2 e^{rt}$

$2r^2 - 4r + 2 = 0$

$2(r-1)^2 = 0$

$\therefore r = 1, 1$

general solution

$\boxed{x(t) = c_1 e^t + c_2 te^t}$

$y = 2x'(t)$

$\boxed{y(t) = 2c_1 e^t + 2c_2 (e^t + te^t)}$

$x(0) = 2 \Rightarrow 2 = c_1$

$y(0) = 4 \Rightarrow -4 = 2c_1 + 2c_1$

$c_1 = 2,$

$c_2 = -4$

$$\boxed{\begin{aligned} x(t) &= 2e^t + (-4)te^t \\ y(t) &= 4e^t - 8(e^t + te^t) \end{aligned}}$$

4. (20 points) Calculate the general solution for

$$y^{(4)} - 8y'' + 16y = e^{-x}$$

Hint: use the quadratic formula with r^2 as your variable. Remember that $r^4 = (r^2)^2$

① Find y_c, the general solution of the homogeneous part

$$y^{(4)} - 8y'' + 16y = 0$$

$y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2e^{rx}$

$y^{(4)} = r^4e^{rx}$

$$r^4 - 8r^2 + 16 = 0$$

$$(r^2)^2 - 8r^2 + 16 = 0$$

$$r^2 = \frac{8 \pm \sqrt{64 - 64}}{2}$$

$$r^2 = 4, 4$$

$$r = \pm 2, \pm 2$$

$$\boxed{y_c = C_1 e^{2x} + C_2 e^{-2x} + C_3 x e^{2x} + C_4 x e^{-2x}}$$

② Find y_p, a particular solution to $y^{(4)} - 8y'' + 16y = e^{-x}$

By undetermined coefficients, we guess $y = Ae^{-x}$

$y'' = Ae^{-x}$

$y^{(4)} = Ae^{-x}$

$$Ae^{-x} - 8Ae^{-x} + 16Ae^{-x} = e^{-x} \Rightarrow 9Ae^{-x} = e^{-x} \Rightarrow \boxed{A = \frac{1}{9}}$$

$$\boxed{y_p = \frac{1}{9}e^{-x}}$$

③ General solution

$$y = y_p + y_c$$

$$= \frac{1}{9}e^{-x} + C_1 e^{2x} + C_2 e^{-2x} + C_3 x e^{2x} + C_4 x e^{-2x}$$

5. (20 points) Consider the boundary value problem

$$y'' + \lambda y = 0,$$

For each of the following boundary conditions, state if $\lambda = 0$ an eigenvalue and explain your answers. If it is an eigenvalue, write down a corresponding eigenfunction.

(a) $y(0) = 0, y(1) = 0.$

(b) $y'(-1) = 0, y'(1) = 0.$

(c) $y'(-\pi) = 0, y(\pi) = 0.$

Solution:

A

For all three cases,

$y = Ax + B$

(a) $B = 0$, $A + B = 0 \Rightarrow A = 0$ $B = 0$

so $y = 0$, and $\lambda = 0$ is not an eigenvalue

(b) $A = 0$, but B can be anything

so $\lambda = 0$ is an eigenfunction, with $y(x) = 7$ an eigenfunction

(c) ~~AE~~ $A = 0$, $A\pi + B = 0.$

$\Rightarrow A = B = 0$

$\lambda = 0$ is not an eigenfunction

6. (10 points) Gideon wants to find the general solution of the differential equation

$$(\sin(x))^2 y''(x) - \sin(x)\cos(x) y'(x) + y(x) = 0.$$

Gideon finds the solutions $y(x) = \sin(x)$ and $y(x) = \sin(-x)$. Since the differential equation is second order, linear and homogeneous, Gideon uses the principle of superposition to determine that the general solution must be

$$y(x) = C_1 \sin(x) + C_2 \sin(-x), \tag{1}$$

for arbitrary constants C_1, C_2.

Gideon's friend Kiora tells Gideon that his general solution must be wrong, because she calculated a solution

$$y(x) = \sin(x) \log\left(\frac{\sin(x/2)}{\cos(x/2)}\right). \tag{2}$$

Kiora says that there are no constants C_1, C_2 that will give Gideon her solution. Using Wolfram Alpha, Gideon checks that Kiora's solution (2) is indeed correct.

Explain why Gideon's general solution was wrong.

For partial credit, you may instead write down the defnitions of "linear", "homogeneous" and the principle of superposition.

sin(x) and sin(-x) are linearly dependent.

Since $\sin(-x) = -\sin(x)$,

so $\dfrac{\sin(x)}{\sin(-x)} = -1$, a constant