MATH 4163-002 Test I

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February 16, 2015 1:30pm-2:45pm

Answer the questions in the spaces provided on the question sheets. No calculators allowed.

Name: ____

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	

1. (10 points) Fourier's law is written as

$$\Phi(x,t) = -K\frac{\partial u(x,t)}{\partial x}.$$

Where $u(x,t)$ is the temperature and $\Phi(x,t)$ is the flux, the rate at which heat energy flows to the right. K is a positive constant. Explain the minus sign in this equation.

Solution:

If $\frac{\partial u(x,t)}{\partial x}$ is positive, the temperature graph has a positive slope and this implies that the right side of the rod is warmer than the left side. Since heat moves from warm to cold areas, the heat flow at this point must go towards the left, which implies that $\Phi(x,t)$ is negative.

If $\frac{\partial u(x,t)}{\partial x}$ is negative, the temperature graph has a negative slope and this implies that the right side of the rod is colder than the left side. Since heat moves from warm to cold areas, the heat flow at this point must go towards the right, which implies that $\Phi(x,t)$ is positive.

2. (15 points) Calculate the steady-state temperature for a 1-dimensional rod of length 5 meters, where the temperature of the left end is set at 10 degrees Celsius for all time, and the right end is perfectly insulated.

Solution:

We have

$$\frac{\partial u(x)}{\partial x^2} = 0, u(0) = 10, u'(5) = 0.$$

$u''(x) = 0$ implies $u(x) = Ax + B$ for constants A, B. The first initial condition implies $B = u(0) = 10$, and the second initial condition implies that $A = u'(5) = 0$. Thus we have $u(x) = 10$, and the steady state solution is a constant 10 degress celsius on the rod.

3. (15 points) Derive the heat equation for a 1-dimensional rod,

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2},$$

Using Fourier's law (given in Problem 1) and the conservation of energy equation. Here $u(x,t)$ is the temperature of the rod at location x and time t.

Solution: Fourier's law is

$$\Phi(x,t) = -K\frac{\partial u(x,t)}{\partial x},$$

where Φ is heat flux.

Temperature and energy density are related as follows:

$$c\rho u(x,t) = e(x,t),$$

where c is specific heat and ρ is mass density.

The energy conservation formula is

$$\begin{aligned}
\frac{\partial e(x,t)}{\partial t} &= -\frac{\partial \Phi(x,t)}{\partial x} \\
\frac{\partial c\rho u(x,t)}{\partial t} &= -\frac{\partial\left(-K\frac{\partial u(x,t)}{\partial x}\right)}{\partial x} \\
\frac{\partial u(x,t)}{\partial t} &= \frac{K}{c\rho}\frac{\partial^2 u(x,t)}{\partial x^2},
\end{aligned}$$

and so once we define $k = K/c\rho$ we are done.

4. (15 points) Calculate the solution to the heat equation,

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2},$$

with boundary conditions $\dfrac{\partial u(0,t)}{\partial x} = \dfrac{\partial u(6,t)}{\partial x} = 0$,

and initial condition

$$u(x,0) = 2 - 3\cos\left(\frac{7\pi x}{6}\right).$$

You may use the following facts: the general solution to the ODE $y'(t) + \lambda k y(t) = 0$ is $y(t) = Ce^{-\lambda kt}$, and the boundary value problem $y''(x) + \lambda y(x) = 0$, $y'(0) = 0, y'(L) = 0$ has eigenvalues $\left(\frac{n\pi}{L}\right)^2$ with corresponding eigenfunction $\cos\left(\frac{n\pi x}{L}\right)$ for $n = 0, 1, 2, \ldots$. Writing $u(x,t)$ as $u(x,t) = F(x)G(t)$ will result in

$$\frac{1}{F(x)}F''(x) = \frac{1}{kG(t)}G'(t) = -\lambda.$$

Solution: We have boundary conditions $F'(0) = F'(6) = 0$, which implies we have eigenvalues $\lambda = \left(\frac{n\pi}{6}\right)^2$ with eigenfunctions $F(x) = \cos(n\pi x/6)$ for $F(x)$. The solution for the $G(t)$ differential equation is $G(t) = Ce^{-\lambda kt}$.

Thus we have product solutions

$$u_n(x,t) = \cos(n\pi x/6)e^{-\left(\frac{n\pi}{6}\right)^2 kt},$$

for $n = 0, 1, 2, \ldots$.

In particular, a general solution to $u(x,t)$ can be written as

$$u(x,t) = \sum_{n=0}^{\infty} A_n u_n(x,t),$$

for some constants A_n.

But we know that $u(x,0) = 2 - 3\cos(7\pi x/6)$. Since $u_n(x,0) = \cos(n\pi x/6)$, this implies that $A_0 = 2$, $A_7 = -3$, and all other A_j are zero. Thus

$$u(x,t) = 2 - 3\cos(7\pi x/6)e^{-\left(\frac{7\pi}{6}\right)^2 kt}.$$

5. (15 points) Let $n, m > 0$. Prove the following trigonometric fact:

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0 \text{ whenever } m \neq n. \\ \frac{L}{2} \text{ whenever } m = n. \end{cases}$$

You may use the following trigonometric identity:

$$2\sin(\theta_1)\sin(\theta_2) = \cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2).$$

Solution:

If $n = m$, we have

$$\begin{aligned}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx =& \frac{1}{2}\int_0^L \cos(0) - \cos(2n\pi x/L)dx \\ =& \frac{L}{2} - \frac{1}{2}\int_0^L \cos(2n\pi x/L)dx \\ =& \frac{L}{2} - \frac{L}{4n\pi x}[\sin(2n\pi x/L)]_{x=0}^{x=L} \\ =& \frac{L}{2} - 0 + 0.\end{aligned}$$

If $n \neq m$, we write $p = n + m$ and $q = n - m$. Neither of these can be zero. Thus

$$\begin{aligned}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx =& \frac{1}{2}\int_0^L \cos(p\pi x/L) - \cos(q\pi x/L)dx \\ =& \frac{L}{2p\pi x}[\sin(p\pi x/L)]_{x=0}^{x=L} - \frac{L}{2q\pi x}[\sin(q\pi x/L)]_{x=0}^{x=L} \\ =& 0 - 0 - 0 + 0.\end{aligned}$$

6. (15 points) Consider a 2-dimensional disk of radius 1. Its temperature is at a steady state. You measure the temperature at the center of the disk and find that it is at 0 Kelvin. What is the temperature of the disk at the position $r = 1$, $\theta = \pi/3$ in polar coordinates? Explain your reasoning thoroughly. (Note: 0 Kelvin is absolute zero, the coldest possible temperature anything can achieve.)

Solution: By the mean value theorem, the temperatures on the circumference are the average of the temperature in the center. Thus the average temperature on the circumference is 0 Kelvin. But since it is impossible to get any colder than 0 Kelvin, the only way to get an average temperature of 0 Kelvin is if every point on the circumference is 0 Kelvin, including $r = 1, \theta = \pi/3$.

7. (15 points) List the product solutions of the Laplace equation on a rectangular tile of length L and height H,

$$\frac{\partial u(x,y)}{\partial x^2} + \frac{\partial u(x,y)}{\partial y^2} = 0,$$

with boundary conditions

$$\frac{\partial u(x,0)}{\partial y} = \frac{\partial u(x,H)}{\partial y} = u(0,y) = 0$$
$$u(L,y) = g(y).$$

Hint: one of the "facts" given in Problem 4 is also useful in this problem. You may use it without proof.

Solution: Product solutions are of the form $u(x,y) = F(x)G(y)$. Plugging that in to the Laplace equation, we have $G(y)F''(x) + F(x)G''(y) = 0$. This is equivalent to

$$\frac{1}{F(x)}F''(x) = -\frac{1}{G(y)}G''(y).$$

But the LHS doesn't depend on y, and the RHS doesn't depend on x. Thus neither side depends on y or x, and they must both be constant. Let us call the constant λ. We have reduced our PDEs to two ODEs,

$$G''(y) + \lambda G(y) = 0, F''(x) - \lambda F(x) = 0,$$

with boundary conditions $G'(0) = G'(H) = 0$, and $F(0) = 0$.

The boundary value problem for $G(y)$ is exactly the one stated in Problem 4: we have eigenvalues $\lambda = (n\pi/H)^2$ with eigenfunctions $\cos(n\pi y/H)$ for $n = 0, 1, 2, \ldots$.

For the $F(x)$ equation, we write $F(x) = e^{rx}$. This gives us $r^2 - \lambda = 0$, and so $r = \pm\sqrt{\lambda} = \pm(n\pi/H)$. This gives us a general solution of

$$F(x) = C_1 e^{\frac{n\pi}{H}x} + C_2 e^{-\frac{n\pi}{H}x}.$$

Plugging in the boundary condition $F(0) = 0$ gives us $C_2 = -C_1$, and so we have

$$F(x) = C_1(e^{\frac{n\pi}{H}x} - e^{-\frac{n\pi}{H}x}).$$

Thus our product solutions $u(x,y) = F(x)G(y)$ must be

$$(e^{\frac{n\pi}{H}x} - e^{-\frac{n\pi}{H}x})\cos(n\pi y/H),$$

for $n = 0, 1, 2, \ldots$.

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Grading Key Please note that this grading key is just a rough plan of how I plan to grade this test. I reserve the right to change the grading scale once I am actually grading the test.

1. A grades are for answers that clearly indicate the student understands the role of the minus sign by explaining the physical intuition of the equation correctly. B grades are for vague, non-specific answers that nevertheless suggest the student understands the equation. For example, the student might explain correctly how $u(x,t)$ and $\Phi(x,t)$ are related without directly (or correctly) addressing the minus sign. C grades are for students who know what $e(x,t)$ and $\Phi(x,t)$ are, and can more or less correctly explain what their derivatives mean, but make no further progress in answering the question. D grades are for students who provide a vague or slightly incorrect description of what $u(x,t), \Phi(x,t)$ and their derivatives are.

2. A grades are for correct answers, perhaps with inconsequential errors in calculation. B grades are for answers that set up the problem more or less correctly, but perhaps mess up a step in the solution. C grades are for answers that set up the problem wrongly, but demonstrate that the student knows roughly how to solve a problem of this type. D grades are for a answers that set up the problem in a way that is only partially correct.

3. A grades are for correct answers, perhaps with inconsequential errors in calculation or manipulating constants. B grades are for answers that are mostly correct, but are perhaps missing a step. C grades are for a correct list of all the equations that lead to the heat equation but without a description of how to put them together. D grades are for a list of equations that lead to the heat equation, but written in a manner that is either incorrect or incomplete.

4. A grades are for correct answers, perhaps with minor calculation errors. B grades are for answers that are more or less correct, except with one of the major steps missing or wrong. C grades are for answers that are partially correct, but with multiple major steps missing or wrong. D grades are for answers that have managed to get at least one significant step right.

5. A grades are for correct answers to both cases, perhaps with minor calculation errors. B grades have serious calculation errors that demonstrate a gap in the student's calculus knowledge, or in understanding the Fourier series calculation. C grades are for answers that set up the problem properly, but fail to make much progress beyond that. D grades are for answers that only set up the problem in a way that is partially correct, without any progress solving for constants.

6. A grades are for correct answers. B grades are for answers that are more or less right, but don't explicitly state the correct mathematical principles. C grades are for answers that either indicate a vague understanding of the physical intuition, or that correctly state the relevant mathematical principles with no other progress. D grade answers contain at least one statement that is correct and relevant in solving the problem.

7. A grades are for correct answers, that perhaps include minor omissions or calculation errors. B grades are for answers that are wrong in a significant way, but nevertheless suggest that the student understands how to find product solutions. C grades are for wrong answers that nevertheless contain multiple significant correct steps. D grades are able to make some progress in starting the problem beyond just re-writing the question.