MATH 3113-170 Test I

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Answer the questions in the spaces provided on the question sheets. No calculators allowed.

Name: ____

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

1. (15 points) By plugging in, check that $y_1(x) = e^{-x}$ and $y_2(x) = xe^{-x}$ are solutions of $y'' + 2y' + y = 0$.

Solution: We have $y_1 = e^{-x}, y_1' = -e^{-x}, y_1'' = e^{-x}$. We check that

$$y_1'' + 2y_1 + y_1 = e^{-x} + 2(-e^{-x}) + e^{-x} = 0.$$

Similarly, we have $y_2 = xe^{-x}, y_2' = e^{-x} - xe^{-x}, y_2'' = -e^{-x} - e^{-x} + xe^{-x}$. We check that

$$-e^{-x} - e^{-x} + xe^{-x} + 2(e^{-x} - xe^{-x}) + xe^{-x} = 0.$$

2. (20 points) A driver slams the brakes on her car when it is going 20 meters per second. Assuming that her brakes provide a constant deceleration of 10 meters per second per second,

   (a) how long does it take for her car to come to a complete stop?

   (b) how far does her car skid before it stops?

Solution: We have a constant deceleration of 10 meters per second per second, so

$$\begin{aligned} x''(t) &= -10, \\ x'(t) &= -10t + C \\ x(t) &= -5t^2 + Ct + D. \end{aligned}$$

Furthermore, we know that $x'(0) = 20$. This tells us that $C = 20$. The car comes to a complete stop at time T where $x'(T) = 0$. So

$$x'(T) = -10T + 20 = 0,$$

which implies $T = 2$, so the car stops after two seconds. Let us assume that the position where the driver slams the brakes is position 0. We have then $x(0) = 0$, and so $D = 0$. We then have $x(2) = -5(2)^2 + 20(2) = 20$, so the car skids for 20 meters before coming to a stop.

3. (20 points) The table below lists values for the continuous function $f(x, y)$.

$y \setminus x$	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**
3	0	1	2	3	4	5	6
2	-1	0	1	2	3	4	5
1	-2	-1	0	1	2	3	4
0	-3	-2	-1	0	1	2	3
-1	-4	-3	-2	-1	0	1	2
-2	-5	-4	-3	-2	-1	0	1
-3	-6	-5	-4	-3	-2	-1	0

Sketch the slope field of the differential equation $y'(x) = f(x, y)$ and the solution curve of the solution with initial condition $x = 2, y = -1$.

Solution: Use the DFIELD applet, using the equation $y'(x) = x + y$ to see a computer-generated plot. Click on the point $(2, -1)$ to see the solution curve.

4. (20 points) Assume that y, x are both positive. Find the general solution for $\frac{dy}{dx} = y/2x$.

Solution: This is a separable equation. So we can rewrite it as

$$\int \frac{1}{y} dy = \int \frac{1}{2x} dx$$

$$\begin{aligned} \ln(y) &= \frac{1}{2}\ln(x) + C \\ e^{\ln(y)} &= e^{\frac{1}{2}\ln(x)+C} \\ y &= e^C\sqrt{x}. \end{aligned}$$

5. (15 points) Find the general solution for

$$x^2y' = xy + y^2.$$

Hint: this is a homogeneous equation, so the substitution $v = y/x$ might be useful.

Solution: We divide the equation by x^2 to obtain

$$y' = \frac{y}{x} + \left(\frac{y}{x}\right)^2.$$

We use the substitution $v = y/x$. We also have $y = vx$, $y' = v + v'x$ to obtain

$$\begin{aligned}
v + v'x =& v + v^2 \\
v'x =& v^2 \\
\frac{dv}{dx} =& \frac{v^2}{x} \\
\int v^{-2} dv =& \int \frac{1}{x} dx \\
-\frac{1}{v} =& \ln|x| + C \\
v =& \frac{-1}{\ln|x| + C} \\
\frac{y}{x} =& \frac{-1}{\ln|x| + C} \\
y =& \frac{-x}{\ln|x| + C}
\end{aligned}$$

We can now solve this equation.

6. (10 points) In order to solve a differential equation

$$y' + P(x)y = Q(x)$$

using the integrating factors method, we first have to multiply the equation by the integrating factor to get

$$y'e^{\int P(x)dx} + P(x)ye^{\int P(x)dx} = Q(x)e^{\int P(x)dx}.$$

Secondly, we can then rewrite the equation as

$$(ye^{\int P(x)dx})' = Q(x)e^{\int P(x)dx}.$$

Explain carefully why the second step is justified.

Solution: We have

$$\begin{aligned}(ye^{\int P(x)dx})' =& y'e^{\int P(x)dx} + y(e^{\int P(x)dx})' \\ =& y'e^{\int P(x)dx} + y(e^{\int P(x)dx})\left(\int P(x)dx\right)' \\ =& y'e^{\int P(x)dx} + y(e^{\int P(x)dx})P(x)\end{aligned}$$

The first equality is due to the product rule, the second is the chain rule, and the third is the fundamental theorem of calculus. We have now shown that the LHS of the last two equations in the question are equivalent.