MATH 3113-009 Test I

Dr. Darren Ong

February 4, 2015 10:30am-11:20am

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. No calculators allowed.

Name: ______________________________

Problem	Score
1	
2	
3	
4	
5	
Total	

1. (20 points) A particle on the real line has acceleration $a(t) = 3t$ meters per second per second. Its initial velocity is -15 meters per second, and its initial position is at the 5 meter mark. Calculate the particle's position after t seconds.

Solution: Since $a(t) = 3t$, we have $v'(t) = 3\frac{t^2}{2}$. Taking the antiderivative with respect to t on both sides, we get the general solution

$$v(t) = 3\frac{t^2}{2} + C,$$

for some arbitrary constant C. Since $v(0) = -15$, we can calculate that

$$-15 = v(0) = 3(0)^2 + C = C,$$

so $C = -15$, and the particular solution for v is

$$v(t) = 3\frac{t^2}{2} - 15.$$

Since $x'(t) = v(t)$, we can rewrite this as

$$x'(t) = 3\frac{t^2}{2} - 15.$$

Again taking the antiderivative with respect to t on both sides, we get

$$x(t) = 3\frac{t^3}{6} - 15t + K,$$

for another arbitrary constant K. We use the initial value $x(0) = 5$ to obtain

$$5 = x(0) = \frac{3}{6}(0)^3 - 15(0) + K = K,$$

and so the particular solution for $x(t)$ is

$$x(t) = 3\frac{t^3}{6} - 15t + 5.$$

2. (20 points) Find the particular solution to the initial value problem

$$\frac{dy}{dx} = ye^{2x}, y(0) = 2e.$$

Solution: This is a separable equation, so we can write

$$\begin{aligned} \frac{1}{y}dy =& e^{2x}dx \\ \int \frac{1}{y}dy =& \int e^{2x}dx \\ \ln|y| =& \frac{e^{2x}}{2} + C \\ |y| =& e^{\frac{e^{2x}}{2}+C}. \end{aligned}$$

Writing $K = \pm e^C$, we have then

$$y = Ke^{\frac{e^{2x}}{2}}.$$

The initial value is $y = 2, x = 0$. Plugging those values in we obtain

$$2 = Ke^{\frac{1}{2}},$$

or in other words, $K = 2/\sqrt{e}$.

The particular solution that satisfies this initial value problem is then

$$y = (2/\sqrt{e})e^{\frac{e^{2x}}{2}}.$$

3. (25 points) Calculate the general solution for the equation

$$y' - 2y = 3e^{2x}.$$

Solution: This is an integrating factor problem. Since the coefficient on the y-term is -2, the integrating factor is

$$e^{\int -2dx} = e^{-2x}.$$

We multiply the equation by the integrating factor:

$$\begin{aligned}
y'e^{-2x} - 2ye^{-2x} =& 3e^{2x}e^{-2x} \\
y'e^{-2x} - 2ye^{-2x} =& 3 \\
(ye^{-2x})' =& 3 \\
\int (ye^{-2x})' dx =& \int 3dx \\
ye^{-2x} =& 3x + C \\
y =& (3x + C)e^{2x}.
\end{aligned}$$

4. (25 points) Find the general solution for $xy'' = y'$. You may assume x, y, and y' are positive.

Solution: We rewrite the equation as

$$x\frac{d^2y}{dx^2} = \frac{dy}{dx}.$$

We choose the substitution

$$u = \frac{dy}{dx}, \frac{du}{dx} = \frac{d^2y}{dx^2}.$$

We can then write

$$x\frac{du}{dx} = u.$$

This is a separable equation. We thus rewrite it as

$$\frac{1}{u}du = \frac{1}{x}dx.$$

Integrating both sides, we find

$$\int \frac{1}{u}du = \int \frac{1}{x}dx.$$

$$\ln(u) = \ln(x) + C.$$

Since we assumed both x and $u = y'$ are positive, we may omit the absolute values in the ln function. Taking exponentials on both sides,

$$\begin{aligned} e^{\ln(u)} &= e^{\ln(x)+C}, \\ u &= xe^C, \\ u &= Kx, \end{aligned}$$

for a constant $K = e^C$. Substituting $u = \dfrac{dy}{dx}$, we get

$$\frac{dy}{dx} = Kx.$$

Taking the antiderivative with respect to x on both sides, we have then

$$y = Kx^2/2 + D,$$

for another constant D.

5. (10 points) Find a function $f(x, y)$ and a point (x_0, y_0) so that $f(x_0, y_0)$ exists, but

$$\frac{dy}{dx} = f(x, y),$$

fails the "existence" part of the existence-uniqueness test around (x_0, y_0). You will get partial credit for a correct statement of the existence-uniqueness theorem.

Solution: The existence part of the existence-uniqueness test says that if $f(x_0, y_0)$ is continuous in any region around (x_0, y_0), then there exists a solution to $y' = f(x, y)$ with initial condition $x = x_0, y = y_0$ in that region.

There are two possible types of solutions to this problem. The first type considers a function $f(x, y)$ that is defined at x_0, y_0, but whose domain does not include some points close to x_0, y_0. For example:

Solution A We consider $f(x, y) = \sqrt{x}$ around the point $(0, 0)$. Here $f(0, 0) = 0$, which exists. The square root function is only defined for non-negative values of x. However, any open interval around $x = 0$ will contain negative values for x, so any region around (0,0) must contain points with negative x-values. On those points, the function $f(x, y)$ is not defined.

Another type of solution simply considers a discontinuous function.

Solution B We consider a function f defined so $f(x, y) = 0$ for $x \leq 0$, but $f(x, y) = 1$ whenever x is positive. We let (x_0, y_0) be $(0, 0)$, and $f(0, 0) = 0$, which exists. It is clear that in any region around that point, $f(x, y)$ has a jump at $x = 0$, so it cannot be continuous. Thus it fails the existence test.